THE MODERN EPISTEMIC INTERPRETATIONS OF PROBABILITY: LOGICISM AND SUBJECTIVISM

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This chapter will focus on the modern epistemic interpretations of probability, namely logicism and subjectivism. The qualification "modern" is meant to oppose the "classical" interpretation of probability developed by Pierre Simon de Laplace (1749-1827). With respect to Laplace's definition, modern epistemic interpretations do not retain the strict linkage with the doctrine of determinism. Moreover, Laplace's "Principle of insufficient reason" by which equal probability is assigned to all possible outcomes of a given experiment (uniform prior distribution) has been called into question by modern epistemic interpretations and gradually superseded by other criteria.

In the following pages the main traits of the logical and subjective interpretations of probability will be outlined together with the position of a number of authors who developed different versions of such viewpoints. The work of Rudolf Carnap, who is widely recognised as the most prominent representative of logicism, will not be dealt with here as it is the topic of another chapter in the present volume.[1]

1 THE LOGICAL INTERPRETATION OF PROBABILITY

1.1 Forefathers

The logical interpretation regards probability as an epistemic notion pertaining to our knowledge of facts rather than to facts themselves. Compared to the "classical" epistemic view of probability forged by Pierre Simon de Laplace, this approach stresses the logical aspect of probability, and regards the theory of probability as part of logic.

According to Ian Hacking, the logical interpretation can be traced back to Leibniz, who entertained the idea of a logic of probability comparable to deductive logic, and regarded probability as a relational notion to be valued in relation to the available data. More particularly, Leibniz is seen by Hacking as anticipating Carnap's programme of inductive logic.[2]

[1]See the chapter by Sandy Zabell in this volume. For a more extensive treatment of the topics discussed here, see Galavotti [2005].

[2]See Hacking [1971] and [1975].

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The idea that probability represents a sort of degree of certainty, more precisely the degree to which a hypothesis is supported by a given amount of information, was later worked out in some detail by the Czech mathematician and logician Bernard Bolzano (1781–1848). Author of the treatise *Wissenschaftslehre* (1837) which is reputed to herald contemporary analytical philosophy,[3] Bolzano defines probability as the "degree of validity" (*Grad der Gültigkeit*) relative to a proposition expressing a hypothesis, with respect to other propositions, expressing the possibilities open to it. Probability is seen as an objective notion, exactly like truth, from which probability derives.[4] The main ingredients of logicism, namely the idea that probability is a logical relation between propositions endowed with an objective character, are found in Bolzano's conception, which can be seen as a direct ancestor of Carnap's theory of probability as *partial implication*.

1.2 Nineteenth century British logicists

In the nineteenth century the interpretation of probability was widely debated in Great Britain, and opposite viewpoints were upheld. Both the empirical and the epistemic views of probability counted followers. The empirical viewpoint imprinted the frequentist interpretation forged by two Cambridge scholars: Robert Leslie Ellis (1817–1859) and John Venn (1834–1923), author of *The Logic of Chance*, that appeared in three editions in 1866, 1876 and 1888. The epistemic viewpoint inspired the logical interpretation embraced by George Boole, Augustus De Morgan and Stanley Jevons, whose work is analysed in volume IV of the *Handbook of the History of Logic*, devoted to *British Logic in the Nineteenth Century*.[5] Therefore, the present account will be limited to a brief outline of these authors' views on probability.

George Boole (1815–1864) is the author of the renowned *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities* (1854). Although his name is mostly associated with (Boolean) algebra, Boole made important contributions to differential and integral calculus, and also probability. According to biographer Desmond MacHale, Boole's work on probability was "greatly encouraged by W.F. Donkin, Savilian Professor of Astronomy in Oxford, who had himself written some important papers on the subject of probability. Boole was gratified that Donkin agreed with his results" [MacHale, 1985, p. 215]. William Donkin, on whom something will be added in the second part of this chapter, shared with Boole an epistemic view of probability, although he was himself closer to the subjective outlook.

According to Boole "probability is expectation founded upon partial knowledge" [Boole, 1854a; 1916, p. 258]. In other words, probability *gives grounds for*

[3]See Dummett [1993].

[4]See Bolzano [1837].

[5]See Gabbay and Woods, eds. [2008], in particular the chapters by Dale Jacquette on "Boole's Logic" (pp. 331–379); Michael E. Hobart and Joan L. Richards on "De Morgan's Logic" (pp. 283–329); and Bert Mosselmans and Ard van Moer on "William Stanley Jevons and the Substitution of Similars" (pp. 515–531).

expectation, based on the information available to those who evaluate it. However, probability is not itself a degree of expectation:

> "The rules which we employ in life-assurance, and in the other statistical applications of the theory of probabilities, are altogether independent of the *mental* phaenomena of expectation. They are founded on the assumption that the future will bear a resemblance to the past; that under the same circumstances the same event will tend to recur with a definite numerical frequency; not upon any attempt to submit to calculation the strength of human hopes and fears". [Boole, 1854a; 1916, pp. 258-259]

Boole summarizes his own attitude thus: "probability I conceive to be not so much expectation, as a rational ground for expectation" [Boole, 1854b; 1952, p. 292]. The accent on rationality features a peculiar trait of the logical interpretation, which takes a normative attitude towards the theory of probability. As we shall see, this marks a major difference from subjectivism.

Within Boole's perspective, the normative character of probability derives from that of logic, to which it belongs. The "laws of thought" investigated in his most famous book are not meant to describe how the mind works, but rather *how it should work* in order to be rational: "the mathematical laws of reasoning are, properly speaking, the laws of *right* reasoning only" [Boole, 1854a, 1916, p. 428].[6] According to Boole's logical perspective, probability does not represent a property of events, being rather a relationship between propositions describing events. In Boole's words:

> "Although the immediate business of the theory of probability is with the frequency of the occurrence of events, and although it therefore borrows some of its elements from the science of number, yet as the expression of the occurrence of those events, and also of their relations, of whatever kind, which connect them, is the office of language, the common instrument of reason, so the theory of probabilities must bear some definite relation to logic. The events of which it takes account are expressed by propositions; their relations are involved in the relations of propositions. Regarded in this light, the object of the theory of probabilities may be thus stated: Given the separate probabilities of any propositions to find the probability of another proposition. By the probability of a proposition, I here mean [...] the probability that in any particular instance, arbitrarily chosen, the event or condition which it affirms will come to pass". [Boole, 1851, 1952, pp. 250-251]

Accordingly, the theory of probability is "coextensive with that of logic, and [...] it recognizes no relations among events but such as are capable of being expressed by propositions" [Boole, 1851, 1952, p. 251].

[6] According to some authors, Boole combines a normative attitude towards logic with psychologism. See Kneale [1948] and the "Introduction" (Part I by Ivor Grattan-Guinness and Part II by Gérard Bornet) in Boole [1997].

Two kinds of objects of interest fall within the realm of probability: games of chance and observable phenomena belonging to the natural and social sciences. Games of chance confront us with a peculiar kind of problems, where the ascertainment of data is in itself a way of measuring probabilities. Events of this kind are called *simple*. Sometimes such events are combined to form a *compound* event, as when it is asked what is the probability of obtaining a six twice in two successive throws of a die. By contrast, the probability of phenomena encountered in nature can only be measured by means of frequencies, and then we face compound events. Simple events are described by simple propositions, and compound events are described by compounded propositions. Simple propositions are combined to form compounded propositions by means of the logical relations of conjunction and disjunction, and the dependence of the occurrence of certain events upon others can be represented by conditional propositions. Once the events subject to probability are described by propositions, these can be handled by using the methods of logic. The fundamental rules for calculating compounded probabilities are presented by Boole in such a way as to show their intimate relation with logic, and more precisely with his algebra. The conclusion attained is that there is a "natural bearing and dependence" [Boole, 1854a, 1916, p. 287] between the numerical measure of probability and the algebraic representation of the values of logical expressions. The task Boole sets himself is

> "to obtain a general method by which, given the probabilities of any events whatsoever, be they simple or compound, dependent or independent, conditioned or not, one can find the probability of some other event connected with them, the connection being either expressed by, or implicit in, a set of data given by logical equations". [Boole, 1854a, 1916, p. 287]

In so doing Boole sets forth the logicist programme, to be resumed by Carnap a hundred years later.[7]

Another representative of nineteenth century logicism is the mathematician Augustus De Morgan (1806–1871), who greatly influenced Boole.[8] De Morgan's major work in logic is the treatise *Formal Logic: or, The Calculus of Inference, Necessary and Probable* (1847) in which he claims that "by degree of probability we really mean, or ought to mean, degree of belief" [De Morgan, 1847, 1926, p. 198]. De Morgan strongly opposed the tenet that probability is an objective feature of objects, like their physical properties: "I throw away objective *probability* altogether, and consider the word as meaning the state of the mind with respect to an assertion, a coming event, or any other matter on which absolute knowledge does not exist" [De Morgan, 1847, 1926, p. 199]. However, when making these claims, De Morgan does not refer to actual belief, entertained by individual persons, but

[7] The reader is addressed to Hailperin [1976] for a detailed account of Boole's theory of probability.

[8] On De Morgan's life see the memoir written by his wife, in De Morgan, Sophia Elizabeth [1882], also containing some correspondence.

rather to the kind of belief a rational agent *ought* adopt when evaluating probability. Therefore, to say that the probability of a certain event is three to one should be taken to mean "that in the universal opinion of those who examine the subject, the state of mind to which a person *ought* to be able to bring himself is to look three times as confidently upon the arrival as upon the non-arrival" [De Morgan, 1847, 1926, p. 200].

De Morgan also wrote some essays specifically devoted to probability, including *Theory of Probabilities* (1837), and *An Essay on Probabilities, and on their Applications to Life, Contingencies and Insurance Offices* (1838), where he maintains that "the quantities which we propose to compare are the forces of the different impressions produced by different circumstances" [De Morgan, 1838, p. 6], and that "probability is the feeling of the mind, not the inherent property of a set of circumstances" [De Morgan, 1838, p. 7]. At first glance, De Morgan's description of probability as a "degree of belief" and "state of mind" associate him with subjectivism. But his insistence upon referring to the human mind as transcending individuals, not to the minds of single agents who evaluate probabilities, sets him apart from modern subjectivists like Bruno de Finetti.

The logicist attitude towards probability also characterizes the work of the economist and logician William Stanley Jevons (1835-1882).[9] In *The Principles of Science* (1873) Jevons claims that "*probability belongs wholly to the mind*" [Jevons, 1873, 1877, p. 198]. While embracing an epistemic approach, Jevons does not define probability as a "degree of belief", because he finds this terminology ambiguous. Against Augustus De Morgan, his teacher at University College London, he maintains that "the nature of *belief* is not more clear [...] than the notion which it is used to define. But an all-sufficient objection is, that *the theory does not measure what the belief is, but what it ought to be*" [Jevons, 1873, 1877, p. 199]. Jevons prefers "to dispense altogether with this obscure word belief, and to say that the theory of probability deals with *quantity of knowledge*" [Jevons, 1873, 1877, p. 199]. So defined, probability is seen as a suitable guide of belief and action. In Jevons' words: "the value of the theory consists in correcting and guiding our belief, and rendering one's states of mind and consequent actions harmonious with our knowledge of exterior conditions" [Jevons, 1873, 1877, p. 199].

Deeply convinced of the utility and power of probability, Jevons established a close link between probability and induction, arguing "that it is impossible to expound the methods of induction in a sound manner, without resting them upon the theory of probability" [Jevons, 1873, 1877, p. 197]. In this connection he praises Bayes' method:

> "No inductive conclusions are more than probable, and [...] the theory of probability is an essential part of logical method, so that the logical value of every inductive result must be determined consciously or unconsciously, according to the principle of the inverse method of probability". [Jevons, 1873, 1877, p. xxix]

[9]See Keynes, [1936, 1972] for a biographical sketch of Jevons.

A controversial aspect of Jevons' work is his defence of Laplace against various criticisms raised by a number of authors including Boole. While granting Laplace's critics that the principle of insufficient reason is to a certain extent arbitrary, he still regards it as the best solution available:

> "It must be allowed that the hypothesis adopted by Laplace is in some degree arbitrary, so that there was some opening for the doubt which Boole has cast upon it. [...] But it may be replied [...] that the supposition of an infinite number of balls treated in the manner of Laplace is less arbitrary and more comprehensive than any other that can be suggested". [Jevons, 1873, 1877, p. 256]

According to Jevons, Laplace's method is of great help in situations characterized by lack of knowledge, so it

> "is only to be accepted in the absence of all better means, but like other results of the calculus of probability, it comes to our aid when knowledge is at an end and ignorance begins, and it prevents us from over-estimating the knowledge we possess". [Jevons, 1873, 1877, p. 269]

When reading Jevons, one is impressed by his deeply probabilistic attitude, testified by statements like the following: "the certainty of our scientific inferences [is] to a great extent a delusion" [Jevons, 1873, 1877, p. xxxi], and "the truth or untruth of a natural law, when carefully investigated, resolves itself into a high or low degree of probability" [Jevons, 1873, 1877, p. 217]. Jevons regards knowledge as intrinsically incomplete and calls attention to the shaky foundation of science, which is based on the assumption of the uniformity of nature. He argues that "those who so frequently use the expression Uniformity of Nature seem to forget that the Universe might exist consistently with the laws of nature in the most different conditions" (Jevons [1873, 1877], p. 749). In view of all this, appeal to probability is mandatory. Although probability does not tell us much about what happens in the short run, it represents our best tool for facing the future:

> "All that the calculus of probability pretends to give, is *the result in the long run*, as it is called, and this really means in *an infinity of cases*. During any finite experience, however long, chances may be against us. Nevertheless the theory is the best guide we can have". [Jevons, 1873, 1877, p. 261]

This suggests that for Jevons the ultimate justification of inductive inference is to be sought on pragmatical grounds.

1.3 William Ernest Johnson

William Ernest Johnson (1858–1931), mathematician, philosopher and logician, Fellow of King's College and lecturer in the University of Cambridge, greatly influenced outstanding personalities such as John Maynard Keynes, Frank Plumpton Ramsey and Harold Jeffreys. His most important work is *Logic*, published in three volumes between 1921 and 1924. By the time of his death he had been working on a fourth volume of *Logic*, dealing with probability. The drafts of the first three chapters were published posthumously in *Mind* in 1932, under the title: "Probability: The Relations of Proposal to Supposal"; "Probability: Axioms" and "Probability: The Deductive and the Inductive Problems". The "Appendix on Eduction", closing the third volume of *Logic*, also focuses on probability.

Johnson adopts a "philosophical" approach to logic, stressing its epistemic aspects. He regards logic as "the analysis and criticism of thought" [Johnson, 1921, 1964, p. xiii], and takes a critical attitude towards formal approaches. By doing so, he set himself apart from the mainstream of the period. In a sympathetic spirit, Keynes observes that Johnson

> "was the first to exercise the epistemic side of logic, the links between logic and the psychology of thought. In a school of thought whose natural leanings were towards formal logic, he was exceptionally well equipped to write formal logic himself and to criticize everything which was being contributed to the subject along formal lines". [Keynes, 1931, 1972, p. 349]

Johnson makes a sharp distinction between "the epistemic aspect of thought", connected with "the variable conditions and capacities for its acquisition", and its "constitutive aspect", referring to "the content of knowledge which has in itself a logically analysable form" [Johnson, 1921, 1964, pp. xxxiii-xxxiv]. The epistemic and grammatical aspects of logic are the two distinct albeit strictly intertwined components along which logic is to be analysed.

Regarding probability, Johnson embraces a logical attitude that attaches probability to propositions. While taking this standpoint, he rejects the conception of probability as a property of events: "familiarly we speak of the probability of an event — he writes — but [...] such an expression is not justifiable" [Johnson, 1932, p. 2]. By contrast,

> "Probability is a character, variable in quantity or degree, which may be predicated of a proposition considered in its relation to some other proposition. The proposition to which the probability is assigned is called the proposal, and the proposition to which the probability of the proposal refers is called the supposal". [Johnson, 1932, p. 8]

The terms "proposal" and "supposal" stand for what are usually called "hypothesis" and "evidence". As Johnson puts it, a peculiar feature of the theory of probability is that when dealing with it "we have to recognise not only the two assertive attitudes of acceptance and rejection of a given assertum, but also a third attitude, in which judgment as to its truth or falsity is suspended; and [...] probability can only be expounded by reference to such an attitude towards a given assertum" [Johnson, 1932, p. 2]. If the act of suspending judgment is a mental fact, and as such is the competence of psychology, the treatment of probability taken in reference to that act is also strongly connected to logic, because logic provides the norms to be imposed on it. The following passage describes in what sense for Johnson probability falls within the realm of logic:

> "The logical treatment of probability is related to the psychological treatment of suspense of judgment in the same way as the logical treatment of the proposition is related to the psychological treatment of belief. Just as logic lays down some conditions for correct belief, so also it lays down conditions for correcting the attitude of suspense of judgment. In both cases we hold that logic is normative, in the sense that it imposes imperatives which have significance only in relation to presumed errors in the processes of thinking: thus, if there are criteria of truth, it is because belief sometimes errs. Similarly, if there are principles for the measurement of probability, it is because the attitude of suspense towards an assertum involves a mental measurable element, which is capable of correction. We therefore put forward the view, that probability is quantitative because there is a quantitative element in the attitude of suspense of judgment". [Johnson, 1932, pp. 2-3]

Johnson distinguished three types of probability statements according to their form. These three types should not be confused, because they give rise to different problems. They are:

> "(1) The singular proposition, *e.g.*, that the next throw will be heads, or that this applicant for insurance will die within a year; (2) The class-fractional proposition, *e.g.*, that, of the applicants to an insurance office, 3/4 of consumptives will die within a year; or that 1/2 of a large number of throws will be heads; (3) The universal proposition, e.g., that all men die before the age of 150 years". [Johnson, 1932, p. 2]

In more familiar terminology, Johnson's worry is to distinguish between propositions referring to (1) a *generic individual* randomly chosen from a population, (2) a *finite sample* or population, (3) an *infinite population*. The distinction is important for both understanding and evaluating statistical inference, and Johnson has the merit of having called attention to it.[10]

Closely related is Johnson's view that probability, conceived as the relation between proposal and supposal, presents two distinct aspects: *constructional* and *inferential*. Grasping the constructional relation between any two given propositions means that "both the form of each proposition taken by itself, and the process by which one proposition is constructed from the other" [Johnson, 1932, p. 4] are taken into account. In the case of probability, the form of the propositions involved and the way in which the proposal is constructed by modification of the

[10]Some remarks on the relevance of the distinction made by Johnson are to be found in Costantini and Galavotti [1987].

supposal will determine the constructional relation between them. On such constructional relation is in turn based the inferential relation, "namely, the measure of probability that should be assigned to the proposal as based upon assurance with respect to the truth of the supposal" [Johnson, 1932, p. 4]. A couple of examples, taken from Johnson's exposition, will illustrate the point:

> "Let the proposal be that 'The next throw of a certain coin will give heads'. Let the supposal be that 'the next throw of the coin will give heads or tails'. Then the relation of probability in which the proposal stands to the supposal is determined by the relation of the predication 'heads' to the predication 'heads or tails'. Or. To take another example, let the proposal be that 'the next man we meet will be tall and red-haired', and the supposal that 'the next man we meet will be tall'. Then the relation of predication 'tall and red-haired' to the predication 'tall' will determine the probability to be assigned to the proposal as depending on the supposal. These two cases illustrate the way in which the logical conjunctions 'or' and 'and' enter into the calculus of probability". [Johnson, 1932, p. 8]

Building on these concepts Johnson developed a theory of logical probability that is ultimately based on a relation of *partial implication* between propositions. This brings Johnson's theory close to Carnap's, with the fundamental difference that Carnap adopted a definition of the "content" of a proposition that relies on the more sophisticated tools of formal semantics.

A major aspect of Johnson's work on probability concerns the introduction of the so-called *Permutation postulate*, which corresponds to the property better known as *exchangeability*. This can be described by saying that exchangeable probability functions assign probability in a way that depends on the number of experienced cases, irrespective of the order in which they have been observed. In other words, under exchangeability probability is invariant with respect to permutation of individuals. This property plays a crucial role within Carnap's inductive logic — where it is named *symmetry* — and de Finetti's subjective theory of probability, which will be examined in the second part of this chapter. As we shall see, Johnson's discovery of this result left some traces in Ramsey's work. Johnson's accomplishment was explicitly acknowledged by the Bayesian statistician Irving John Good, whose monograph *The Estimation of Probabilities. An Essay on Modern Bayesian Methods* opens with the following words: "This monograph is dedicated to William Ernest Johnson, the teacher of John Maynard Keynes and Harold Jeffreys" [Good, 1965, p. v]. In that book Good makes extensive use of what he calls "Johnson's sufficiency postulate", a label that he later modified by substituting the term "sufficiency" with "sufficientness". Sandy Zabell's article "W.E. Johnson's 'Sufficientness' Postulate" offers an accurate reconstruction of Johnson's argument, giving a generalisation of it and calling attention to its relevance for Bayesian statistics.[11]

[11] See Zabell [1982]. Also relevant are Zabell [1988] and [1989]. All three papers are reprinted

By contrast, the insight of Johnson's treatment of probability was not grasped by his contemporaries, and his contribution, including the exchangeability result, remained almost ignored. Charlie Dunbar Broad's comment on Johnson's "Appendix on Eduction" testifies to this attitude: "about the Appendix all I can do is, with the utmost respect to Mr Johnson, to parody Mr Hobbes' remark about the treatises of Milton and Salmasius: 'very good mathematics; I have rarely seen better. And very bad probability; I have rarely seen worse"' [Broad, 1924, p. 379].

1.4 John Maynard Keynes: a logicist with a human face

The economist John Maynard Keynes (1883–1946), one of the leading celebrities of the last century, embraced the logical view in his *A Treatise on Probability* (1921). Son of the logician John Neville Keynes, Maynard was educated at Eton and Cambridge, where he later became a scholar and member of King's College. Besides playing a crucial role in public life as a political advisor, Keynes was an indefatigable supporter of the arts, as testified, among other things, by his contribution to the establishment of the Cambridge Arts Theatre. In "A Cunning Purchase: the Life and Work of Maynard Keynes" Roger Backhouse and Bradley Bateman observe that "Keynes' role as an economic problem-solver and a patron of the arts would continue through his last decade, despite his poor health" (Backhouse and Bateman [2006], p. 4). In Cambridge, Keynes was member of the "Apostles" discussion society — also known as "The Society"[12] — together with personalities of the calibre of Lytton Strachey, Leonard Woolf, Henry Sidgwick, John McTaggart, Alfred North Whitehead, Bertrand Russell, Frank Ramsey and, last but not least, George Edward Moore. The latter exercised a great influence on the group, as well as on the partly overlapping "Bloomsbury group", of which Maynard was also part. It was in this atmosphere deeply imbued with philosophy that the young Keynes wrote his book on probability.

In *The Life of John Maynard Keynes* Roy Forbes Harrod maintains that the *Treatise* was written in the years 1906-1911.[13] Although by that time the book was all but completed, Keynes could not prompt its final revision until 1920, due to his political commitments. When it finally appeared in print in 1921, the book was very well received, partly because of the fame that Keynes had by that time gained as an economist and political adviser, partly because it was the first systematic work on probability by an English writer after John Venn's *The Logic of Chance*, whose first edition had been published forty-five years earlier, namely in 1866. A review of the *Treatise* by Charlie Dunbar Broad opens with this passage: "Mr Keynes' long awaited work on Probability is now published, and will at once take its place as the best treatise on the logical foundations of the subject" [Broad, 1922, p. 72], and closes as follows: "I can only conclude by congratulating Mr Keynes

in Zabell [2005].

[12]See Levy [1979] and Harrod [1951] for more details on the Apostles Society.

[13]See Harrod [1951]. On Keynes' life see also Skidelsky [1983-1992].

on finding time, amidst so many public duties, to complete this book, and the philosophical public on getting the best work on Probability that they are likely to see in this generation" [Broad, 1922, p. 85]. Referring to this statement, in an obituary of Keynes Richard Bevan Braithwaite observes that "Broad's prophecy has proved correct" [Braithwaite, 1946, p. 284]. More evidence of the success attained by the *Treatise* is offered by Braithwaite in the portrait "Keynes as a Philosopher", included in the collection *Essays on John Maynard Keynes*, edited by Maynard's nephew Milo Keynes, where he maintains that

> "The *Treatise* was enthusiastically received by philosophers in the empiricist tradition. [...] The welcome given to Keynes' book was largely due to the fact that his doctrine of probability filled an obvious gap in the empiricist theory of knowledge. Empiricists had divided knowledge into that which is 'intuitive' and that which is 'derivative' (to use Russell's terms), and he regarded the latter as being passed upon the former by virtue of there being a logical relationships between them. Keynes extended the notion of logical relation to include probability relations, which enabled a similar account to be given of how intuitive knowledge could form the basis for rational belief which fell short of knowledge". [Braithwaite, 1975, pp. 237-238]

Braithwaite's remarks remind us that at the time when Keynes' *Treatise* was published, empiricist philosophers, under the spell of works like Russell's and Whitehead's *Principia Mathematica*, paid more attention to the deductive aspects of knowledge than to probability. Nevertheless, one should not forget that, as we have seen, the logical approach to probability already counted a number of supporters in Great Britain. Besides, in the same years a similar approach was embraced in Austria by Ludwig Wittgenstein and Friedrich Waismann.[14]

In the "Preface" to the *Treatise* Keynes acknowledges his debt to William Ernest Johnson, and more generally to the Cambridge philosophical setting, regarded as an ideal continuation of the great empiricist tradition "of Locke and Berkeley and Hume, of Mill and Sidgwick, who, in spite of their divergencies of doctrine, are united in a preference for what is matter of fact, and have conceived their subject as a branch rather of science than of creative imagination" [Keynes, 1921, pp. v-vi].

Keynes takes the theory of probability to be a branch of logic, more precisely that part of logic dealing with arguments that are not conclusive, but can be said to have a greater or lesser degree of inconclusiveness. In Keynes' words: "Part of our knowledge we obtain direct; and part by argument. The Theory of Probability is concerned with that part which we obtain by argument, and treats of the different degrees in which the results so obtained are conclusive or inconclusive" [Keynes, 1921, p. 3]. Like the logic of conclusive arguments, the logic of probability investigates the general principles of inconclusive arguments. Both certainty and probability depend on the amount of knowledge that the premisses of

[14]See Galavotti [2005], Chapter 6, for more on Wittgenstein and Waismann.

an argument convey to support the conclusion, the difference being that certainty obtains when the amount of available knowledge authorizes full belief, while in all other cases one obtains degrees of belief. Certainty is therefore seen as the limiting case of probability.

While regarding probability as the expression of partial belief, or degree of belief, Keynes points out that it is an intrinsically relational notion, because it depends on the information available:

> "The terms *certain* and *probable* describe the various degrees of rational belief about a proposition which different amounts of knowledge authorize us to entertain. All propositions are true or false, but the knowledge we have of them depends on our circumstances; and while it is often convenient to speak of propositions as certain or probable, this expresses strictly a relationship in which they stand to a *corpus* of knowledge, actual or hypothetical, and not a characteristic of the propositions in themselves. A proposition is capable at the same time of varying degrees of this relationship, depending upon the knowledge to which it is related, so that it is without significance to call a proposition probable unless we specify the knowledge to which we are relating it". [Keynes, 1921, pp. 3-4]

Another passage states the same idea even more plainly: "No proposition is in itself either probable or improbable, just as no place can be intrinsically distant; and the probability of the same statement varies with the evidence presented, which is, as it were, its origin of reference" [Keynes, 1921, p. 7].

The *corpus* of knowledge on which probability assessments are based is described by a set of propositions that constitute the premisses of an argument, standing in a logical relationship with the conclusion, which describes a hypothesis. Probability resides in this logical relationship, and its value is determined by the information conveyed by the premisses of the arguments involved:

> "As our knowledge or our hypothesis changes, our conclusions have new probabilities, not in themselves, but relatively to these new premisses. New logical relations have now become important, namely those between the conclusions which we are investigating and our new assumptions; but the old relations between the conclusions and the former assumptions still exist and are just as real as these new ones" [Keynes, 1921, p. 7]

On this basis, Keynes developed a theory of comparative probability, in which conditional probabilities are ordered in terms of a relation of "more" or "less" probable, and are combined into compound probabilities.

Like Boole, Keynes aimed to develop a theory of the reasonableness of degrees of belief on logical grounds. Within his perspective the logical character of probability goes hand in hand with its *rational* character. This element is pointed out by Keynes, who maintains that the theory of probability as a logical relation "is concerned with the degree of belief which it is *rational* to entertain in given conditions, and not merely with the actual beliefs of particular individuals, which may or may not be rational" [Keynes, 1921, p. 4]. In other words, Keynes' logical interpretation gives the theory of probability a normative value: "we assert that we *ought* on the evidence to prefer such and such a belief. We claim rational grounds for assertions which are not conclusively demonstrated" [Keynes, 1921, p. 5]. The kernel of the logical interpretation of probability lies precisely with the idea Keynes states with great clarity that in the light of the same amount of information the logical relation representing probability is the same for anyone. So conceived probability is *objective*, its objectivity being warranted by its logical character:

> "What particular propositions we select as the premisses of *our* argument naturally depends on subjective factors peculiar to ourselves; but the relations, in which other propositions stand to these, and which entitle us to probable beliefs, are objective and logical". [Keynes, 1921, p. 4]

It is precisely because the logical relations between the premisses and the conclusion of inconclusive arguments provide objective grounds for belief, that belief based on them can qualify as *rational*.

As to the character of the logical relations themselves, Keynes says that "we cannot analyse the probability-relation in terms of simpler ideas" [Keynes, 1921, p. 8]. They are therefore taken as primitive, and their justification is left to our intuition. Keynes' conception of the objectivity of probability relations and his use of intuition in that connection have been ascribed by a number of authors to Moore's influence. Commenting on Keynes' claim that "what is probable or improbable" in the light of a give amount of information is "fixed objectively, and is independent of our opinion" [Keynes, 1921, p. 4], Donald Gillies observes that when "Keynes speaks of probabilities as being fixed objectively [...] he means objective in the Platonic sense, referring to something in a supposed Platonic world of abstract ideas", and adds that

> "we can see here clearly the influence of G. E. Moore. [...] In fact, there is a very notable similarity between the Platonic world as postulated by Cambridge philosophers in the Edwardian era and the Platonic world as originally described by Plato. Plato's world of objective ideas contained the ethical qualities with the idea of the Good holding the principal place, but it also contained mathematical objects. The Cambridge philosophers thought that they had reduced mathematics to logic. So their Platonic world contained, as well as ethical qualities such as 'good', logical relations". [Gillies, 2000, p. 33]

The attitude just described is responsible for a most controversial feature of Keynes' theory, namely his tenet that probability relations are not always measurable, nor comparable. He writes:

> "By saying that not all probabilities are measurable, I mean that it is not possible to say of every pair of conclusions, about which we have some knowledge, that the degree of our rational belief in one bears any numerical relation to the degree of our rational belief in the other; and by saying that not all probabilities are comparable in respect of more and less, I mean that it is not always possible to say that the degree of our rational belief in one conclusion is either equal to, greater than, or less than the degree of our belief in another". [Keynes, 1921, p. 34]

In other words, Keynes admits of some probability relations which are intractable by the calculus of probabilities. Far from being worrying to him, this aspect testifies to the high value attached by Keynes to intuition. On the same basis, Keynes is suspicious of a purely formal treatment of probability, and of the adoption of mechanical rules for the evaluation of probability.

Keynes believes that measurement of probability rests on the equidistribution of priors: "In order that numerical measurement may be possible, we must be given a number of *equally* probable alternatives" [Keynes, 1921, p. 41]. This admission notwithstanding, Keynes sharply criticizes Laplace's principle of insufficient reason, which he prefers to call "Principle of Indifference" to stress the role of individual judgment in the ascription of *equal* probability to all possible alternatives "if there is an absence of positive ground for assigning *unequal* ones" [Keynes, 1921, p. 42]. To Laplace he objects that "the rule that there must be no ground for preferring one alternative to another, involves, if it is to be a guiding rule at all, and not a *petitio principii*, an appeal to judgments of *irrelevance*" [Keynes, 1921, pp. 54-55]. The judgment of indifference among various alternatives has to be substantiated with the assumption that there could be no further information, on account of which one might change such judgment itself. While in the case of games of chance this kind of assumption can be made without problems, most situations encountered in everyday life are characterized by a complexity that makes it arbitrary. For Keynes, the extension of the principle of insufficient reason to cover all possible applications is the expression of a superficial way of addressing probability, regarded as a product of ignorance rather than knowledge. By contrast, Keynes maintains that the judgment of indifference among available alternatives should not be grounded on ignorance, but rather on knowledge, and recommends that the application of the principle in question always be preceded by an act of discrimination between relevant and irrelevant elements of the available information, and by the decision to neglect certain pieces of evidence.

A most interesting aspect of Keynes' treatment is his discussion of the paradoxes raised by Laplace's principle. As observed by Gillies: "It is greatly to Keynes' credit that, although he advocates the Principle of Indifference, he gives the best statement in the literature [Keynes, 1921, Chapter 4] of the paradoxes to which it gives rise" [Gillies, 2000, p. 37]. The reader is addressed to Chapter 3 of Gillies' *Philosophical Theories of Probability* for a critical account of Keynes' treatment of the matter.

Keynes' distrust in the practice of unrestrictedly applying principles holding within a restricted domain regards not only the Principle of Indifference, but extends to the "Principle of Induction", taken as the method of establishing empirical knowledge from a multitude of observed cases. Keynes is suspicious of the inference of general principles on an inductive basis, including causal laws and the principle of uniformity of nature. He distinguishes two kinds of generalizations "arising out of empirical argument". First, there are universal generalizations corresponding to "*universal induction*", of which he says that "although such inductions are themselves susceptible of any degree of probability, they affirm *invariable* relations". Second, there are those generalizations which assert probable connections, which correspond to "*inductive correlation*" [Keynes, 1921, p. 220]. Both types are discussed at length, the first in Part III of the *Treatise* and the second in Part V. Keynes stresses the importance of the connection between probability and induction, a relationship that was clearly seen by Thomas Bayes and Richard Price in the eighteenth century, but was underrated by subsequent literature. After mentioning Jevons, and also "Laplace and his followers", as representatives of the tendency to use probability to address inductive problems, Keynes adds:

> "But it has been seldom apprehended clearly, either by these writers or by others, that the validity of every induction, strictly interpreted, depends, not on a matter of fact, but on the existence of a relation of probability. An inductive argument affirms, not that a certain matter of fact *is* so, but that *relative to certain evidence* there is a probability in its favour". [Keynes, 1921, p. 221]

In other words, probability is not reducible to an empirical matter:

> "The validity and reasonable nature of inductive generalisation is [...] a question of logic and not of experience, of formal and not of material laws. The actual constitution of the phenomenal universe determines the character of our evidence; but it cannot determine what conclusions *given* evidence *rationally* supports". [Keynes, 1921, p. 221]

Granted that induction has to be based on probability, the objectivity and rationality of probabilistic reasoning rests on the logical character of probability taken as the relation between a proposition expressing a given body of evidence and a proposition expressing a given hypothesis.

On the same basis, Keynes criticizes inferential methods entirely grounded on repeated observations, like the calculation of frequencies. Against this attitude, he claims that the similarities and dissimilarities among events must be carefully considered before quantitative methods can be applied. In this connection, a crucial role is played by *analogy*, which becomes a prerequisite of statistical inductive methods based on frequencies. In Keynes' words:

> "To argue from the *mere* fact that a given event has occurred invariably in a thousand instances under observation, without any analysis of the circumstances accompanying the individual instances, that it is likely

> to occur invariably in future instances, is a feeble inductive argument, because it takes no account of the Analogy". [Keynes, 1921, p. 407]

The insistence upon analogy is a central feature of the perspective taken by Keynes, who devotes Part III of the *Treatise* to "Induction and analogy". In an attempt to provide a logical foundation for analogy, Keynes finds it necessary to assume that the variety encountered in the world has to be of a limited kind:

> "As a logical foundation for Analogy, [...] we seem to need some such assumption as that the amount of variety in the universe is limited in such a way that there is no one object so complex that its qualities fall into an infinite number of independent groups (i.e., groups which might exist independently as well as in conjunction); or rather that none of the objects about which we generalise are as complex as this; or at least that, though some objects may be infinitely complex, we sometimes have a finite probability that an object about which we seek to generalise is not infinitely complex". [Keynes, 1921, p. 258]

This assumption confers a finitistic character to Keynes' approach, criticized, among others, by Rudolf Carnap.[15] The principle of limited variety is attacked on a different basis by Ramsey. The topic is addressed in two notes included in the collection *Notes on Philosophy, Probability and Mathematics*, namely "On the Hypothesis of Limited Variety", and "Induction: Keynes and Wittgenstein", where Ramsey claims to see "no logical reason for believing any such hypotheses; they are not the sort of things of which we could be supposed to have a priori knowledge, for they are complicated generalizations about the world which evidently may not be true" [Ramsey, 1991a, p. 297].

Another important ingredient of Keynes' theory is the notion of *weight* of arguments. Like probability, the weight of inductive arguments varies according to the amount of evidence. But while probability is affected by the proportion between favourable and unfavourable evidence, weight increases as relevant evidence, taken as the sum of positive and negative observations, increases. In Keynes' words:

> "As the relevant evidence at our disposal increases, the magnitude of the probability of the argument may either decrease or increase, according as the new knowledge strengthens the unfavourable or the favourable evidence; but *something* seems to have increased in either case — we have a more substantial basis upon which to rest our conclusion. I express this by saying that an accession of new evidence increases the *weight* of an argument. New evidence will sometimes decrease the probability of an argument, but it will always increase its 'weight"'. [Keynes, 1921, p. 71]

The concept of weight mingles with that of relevance, because to say that a piece of evidence is relevant is the same as saying that it increases the weight of an

[15] See Carnap [1950], § 62.

argument. Therefore, Keynes' stress on weight backs the importance of the notion of relevance within his theory of probability.

Keynes addresses the issue of whether the weight of arguments should be made to bear upon action choice. As he puts it: "the question comes to this — if two probabilities are equal in degree, ought we, in choosing our course of action, to prefer that one which is based on a greater body of knowledge?" [Keynes, 1921, p. 313]. This issue, he claims, has been neglected by the literature on action choice, essentially based on the notion of mathematical expectation. However, Keynes admits to find the question "highly perplexing", adding that "it is difficult to say much that is useful about it" [Keynes, 1921, p. 313]. The discussion of these topics leads to a sceptical conclusion, reflecting Keynes' distrust in a strictly mathematical treatment of the matter, motivated by the desire to leave room for individual judgment and intuition. He maintains that:

> "The hope, which sustained many investigators in the course of the nineteenth century, of gradually bringing the moral sciences under the sway of mathematical reasoning, steadily recedes — if we mean, as they meant, by mathematics the introduction of precise numerical methods. The old assumptions, that all quantity is numerical and that all quantitative characteristics are additive, can be no longer sustained. Mathematical reasoning now appears as an aid in its symbolic rather that in its numerical character". [Keynes, 1921, p. 316]

Keynes' notion of weight is the object of a vast literature. Some authors think that such a notion is at odds with the logicist notion of probability put forward by Keynes. For instance, in "Keynes' Theory of Probability and its Relevance to its Economics" Allin Cottrell argues that "the perplexities surrounding 'weight'... are important as the symptom of an *internal* difficulty in the notion of probability Keynes wishes to promote" [Cottrell, 1993, p. 35]. More precisely, Cottrell believes that the idea that some probability judgments are more reliable than others by virtue of being grounded on a larger weight requires that probabilities of probabilities are admitted, while Keynes does not contemplate them. Cottrell thinks that the frequency notion of probability could do the job. As a matter of fact, this clutch of problems has been extensively dealt with by a number of authors operating under the label of "Bayesianism", mostly of subjective orientation.

Keynes' views on the objectivity of probability relations involves the tenet that the validity of inductive arguments cannot be made to depend on their success, and it is not undermined by the fact that some events which have been predicted do not actually take place. Induction allows us to say that on the basis of a certain piece of evidence a certain conclusion is reasonable, not that it is true. Awareness of this fact should inspire caution towards inductive predictions, and Keynes warns against the danger of making predictions obtained by detaching the conclusion of an inductive argument. This features a typical aspect of the logical interpretation of probability, that has been at the centre of a vast debate, in which Rudolf Carnap also took part.[16]

The refusal of any attempt to ground probabilistic inference on success, that goes along with Keynes' insistence on the logical and non-empirical character of probability relations, is stressed by Anna Carabelli, who writes that

> "Keynes was [...] critical of the positivist *a posteriori* criterion of the validity of induction, by which the inductive generalization was valid as far as the prevision based on it will prove successful, that is, will be confirmed by subsequent facts. [...] On the contrary, the validity of inductive method, according to Keynes, did not depend on the success of its prediction, or on its empirical confirmation". [Carabelli, 1988, p. 66]

However, Carabelli adds that "notably, that was what made the difference between Keynes' position and that of those later logico-empiricists, like R. Carnap, who analysed induction from what he called the 'confirmation theory' point of view" [Carabelli, 1988, p. 66]. This claim is misleading, because Carnap's confirmation theory is not so closely linked to the criterion of success as Carabelli claims. In his late writings Carnap appealed to "inductive intuition" to justify induction, thereby embracing a position not so distant from that of Keynes.[17] But it should be added that Keynes assigns to intuition a much more substantial role than Carnap. A further difference between these two authors amounts to their different attitude towards formalization. Keynes, as we have seen, distrusted the pervasive use of mathematics and formal methods, whereas Carnap embraced a strictly formal approach. Carnap's production on probability, which culminated with the publication in 1950 of *Logical Foundations of Probability* and occupied the last twenty years of his life, until he died in 1970, is no exception. Although his perspective underwent significant changes, Carnap never abandoned the programme of developing an inductive logic aimed at providing a rational reconstruction of probability within a formalized logical system. As described by Richard Jeffrey's colourful expression, Carnap "died with his logical boots on, at work on the project" [Jeffrey, 1991, p. 259]. In this enterprise, Carnap was inspired by an unwavering faith in the powers of formal logic on the one side, and of experience on the other, in compliance with the logical empiricist creed. By contrast, Keynes embraced a moderate version of logicism, a logicism "with a human face", imbued with a deeply felt need not to lose sight of ordinary speech and practice, and to assign an essential role to intuition and individual judgment.

To conclude this presentation of Keynes' views on probability, it is worth mentioning the long debated issue of Ramsey's criticism and Keynes' reaction to it. Soon after the publication of the *Treatise*, Ramsey published a critical review in *The Cambridge Magazine* challenging some of the central issues in the *Treatise*, like the conviction that there are unknown probabilities, the principle of limited

[16] See, for instance, Kyburg [1968] and the discussion following it, with comments by Y. Bar-Hillel, P. Suppes, K.R. Popper, W.C. Salmon, J. Hintikka, R. Carnap, H. Kyburg jr.

[17] See Carnap [1968].

variety, and the very idea that probability is a logical relation.[18] As will be argued in more detail in what follows, Ramsey is very critical of this point, which also reappears in other writings. For instance, in "Truth and Probability" he objects to Keynes that "there really do not seem to be any such things as the probability relations he describes" [Ramsey, 1990a, p. 57], and in another note called "Criticism of Keynes" he maintains that: "there are no such things as these relations" [Ramsey, 1991a, p. 273].

After Ramsey's premature death in 1930, Keynes wrote an obituary containing an explicit concession to Ramsey's criticism. There he writes:

> "Ramsey argues, as against the view which I had put forward, that probability is concerned not with objective relations between propositions but (in some sense) with degrees of belief, and he succeeds in showing that the calculus of probabilities simply amounts to a set of rules for ensuring that the system of degrees of belief which we hold shall be a *consistent* system. Thus the calculus of probabilities belongs to formal logic. But the basis of our degrees of belief — or the *a priori* probabilities, as they used to be called — is part of our human outfit, perhaps given us merely by natural selection, analogous to our perceptions and our memories rather than to formal logic. So far I yield to Ramsey — I think he is right". [Keynes, 1930, 1972, p. 339]

Before adding some comments, it is worth recalling how the above quoted passage continues:

> "But in attempting to distinguish 'rational' degrees of belief from belief in general he [Ramsey] was not yet, I think, quite successful. It is not getting to the bottom of the principle of induction merely to say that it is a useful mental habit". [Keynes, 1930, 1972, p. 339]

As one can see, some ten years after publication of the *Treatise*, Keynes was still concerned with drawing a sharp boundary between *rational belief* and *actual belief*. Undeniably, such an attitude sides him with logicism, as opposed to subjectivism. indexlogicism

The literature is divided among those who believe that after Ramsey's criticism Keynes changed his attitude towards probability, and those who are instead convinced that Keynes never changed his mind in a substantial way. Among others, Anna Carabelli believes that "Keynes did not change substantially his view on probability" [Carabelli, 1988, p. 255]. By contrast, Bradley Bateman in "Keynes' Changing Conception of Probability" holds that the views on probability retained in the *Treatise* "underwent at least two significant changes in subsequent years. Keynes first advocated an objective epistemic theory of probability, but later advocated both subjective epistemic and objective aleatory theories of probability" [Bateman, 1987, p. 113]. A still different viewpoint is taken by Donald Gillies,

[18]See Ramsey [1922].

who agrees with Bateman that Keynes changed his conception of probability as a consequence of Ramsey's criticism, but disagrees as to the nature of such change. Gillies argues that

> "Keynes did realize, in the light of Ramsey's criticism, that his earlier views on probability needed to be changed, and he may well have had some rough ideas about how this should be done, but he never settled down to work out a new interpretation of probability in detail. What we have to do therefore is not so much try to reconstruct, on the basis of rather fragmentary evidence, Keynes' exact views on probability in the 1930s. I don't believe that Keynes had very exact views on probability at that time. I suggest therefore that we should switch to trying to develop an interpretation of probability that fits the economic theory that Keynes presented in 1936 and 1937, but without necessarily claiming that his theory was precisely what Keynes himself had in mind". [Gillies, 2006, p. 210]

Keynes' works to which Gillies refers are the well known book *The General Theory of Employment, Interest and Money* published in 1936, and the article "The General Theory of Employment" published in 1937. According to Gillies, Keynes accepted Ramsey's criticisms to some extent, and moved to a theory of probability that he labels "intersubjective" and describes as intermediate between logicism and subjectivism. Its distinctive feature is that of ascribing degrees of belief not to single individuals, as subjectivists do, but rather to groups. Gillies presents the theory as an extension of the subjective viewpoint, by demonstrating a Dutch Book Theorem holding for groups, which shows the following:

> "Let B be some social group. Then it is the interest of B as a whole if its members agree, perhaps as a result of rational discussion, on a common betting quotient rather than each member of the group choosing his or her own betting quotient. If a group does in fact agree on a common betting quotient, this will be called the *intersubjective* or *consensus* probability of the social group". [Gillies, 2006, p. 212][19]

Gillies argues that this interpretation "fits perfectly with Keynes' theory of long-term expectation developed in his 1936 and 1937 publications" [Gillies, 2006, p. 212].

The issue of Keynes' reaction to Ramsey's criticisms and the relationship between his conception of probability and his views on economic theory is the object of ongoing debate.

1.5 Harold Jeffreys between logicism and subjectivism

Professor of astronomy and experimental philosophy at Cambridge University, Harold Jeffreys (1891–1989) is reputedly one of the last century's most prominent

[19]See also Gillies [2000], Chapter 8, for more on the intersubjective theory of probability.

geophysicists and a pioneer of the study of the Earth. As described by Alan Cook in a memoir of Jeffreys written for the Royal Society, "the major spherically symmetrical elements of the structure of the Earth that he [Jeffreys] did so much to elucidate, are the basis for all subsequent elaboration, and generations of students learnt their geophysics from his book *The Earth*" [Cook, 1990, p. 303]. Jeffreys' work also left a mark in other fields, like seismology and meteorology, and, last but not least, probability. [20] His interest in probability and scientific method led to publication of the book *Scientific Inference* in 1931, followed in 1939 by *Theory of Probability*. In addition, he published a number of articles on the topic.

Jeffreys was a wholehearted inductivist who used to say that Bayes' theorem "is to the theory of probability what Pythagoras' theorem is to geometry" [Jeffreys, 1931, p. 7]. He was led to embrace Bayesianism by his own work in geophysics, where he only had access to scarce data, and needed a method for assessing hypotheses regarding unknown situations, like the composition of the Earth. As a practising scientist he was faced with problems of inverse probability, having to explain experimental data by means of different hypotheses, or to evaluate general hypotheses in the light of changing data. This made it natural for Jeffreys to adopt both an epistemic notion of probability and Bayesian methodology, although at the time he started working on this kind of problems Bayesian method was in disgrace among scientists and statisticians, for the most part supporters of frequentism. But as David Howie observed, "restricted to repeated sampling from a well-behaved population, and largely reserved for data reduction" frequentism "could apply neither to the diverse pool of data Jeffreys drew upon nor directly to the sorts of questions he was attempting to address" [Howie, 2002, p. 127].

Jeffreys' refusal to embrace frequentism is responsible for the fact that his contribution to probability was not fully appreciated by his contemporaries, and he engaged a debate with various authors, including the physicist Norman Campbell, and the statistician Ronald Fisher.[21] Against frequentism, Jeffreys holds that "no 'objective' definition of probability in terms of actual or possible observations, or possible properties of the world, is admissible" [Jeffreys, 1939, 1961, p. 11].[22] As will be argued in what follows, this relationship is actually reversed within Jeffreys' epistemology, where probability comes before the notions of objectivity, reality and the external world [Jeffreys, 1936a, p. 325].

Jeffreys started working on probability together with Dorothy Wrinch, a mathematician and scientist, at the time fellow of Girton College Cambridge, who had approached epistemological questions under the influence of Johnson and Russell. In three papers written between 1919 and 1923[23] Jeffreys and Wrinch draw the lines of an inductivist programme that Jeffreys kept firm throughout his long life, and put at the core of a genuinely probabilistic epistemology revolving around the

[20]For a scientific portrait of Harold Jeffreys centred on probability and statistics see Lindley [1991].

[21]See Howie [2002] for a detailed reconstruction of the genesis of Jeffreys' Bayesianism and the polemics he entertained with Fisher.

[22]In connection with Jeffreys' criticism of frequentism see also Jeffreys [1933] and [1934].

[23]See Jeffreys and Wrinch [1919], [1921] and [1923].

idea that probability is "the most fundamental and general guiding principle of the whole of science" [Jeffreys, 1931, p. 7].

Jeffreys and Wrinch made the assumption that all quantitative laws form an enumerable set, and their probabilities form a convergent series. This assumption allows for the assignment of significant prior probabilities to general hypotheses. In addition, Jeffreys and Wrinch formulated a *simplicity postulate*, according to which simpler laws are assigned a greater prior probability.[24] According to its proponents, this principle corresponds to the practice of testing possible laws in order of decreasing simplicity. This machinery allows for the adoption of Bayesian method.

Jeffreys' inductivism is grounded in an epistemic view of probability that shares the main features of logicism, but in certain respects comes closer to subjectivism. According to Jeffreys probability "expresses a relation between a proposition and a set of data" [Jeffreys, 1931, p. 9]. Probability is deemed "a purely epistemological notion" [Jeffreys, 1955, p. 283], corresponding to the *reasonable* degree of belief that is warranted by a certain body of evidence, by which it is uniquely determined. Given a set of data, Jeffreys claims, "a proposition q has in relation to these data one and only one probability. If any person assigns a different probability, he is simply wrong" [Jeffreys, 1931, p. 10]. The conviction that there exists "unique reasonable degrees of belief" [Jeffreys, 1939, p. 36] puts him in line with logicism, while marking a crucial divergence from subjectivism, a divergence described by Bruno de Finetti as that between "necessarists" who affirm and subjectivists who deny "that there are logical grounds for picking out *one single* evaluation of probability as being *objectively* special and 'correct"' [de Finetti, 1970, English edition 1975, vol. 2, p. 40].

For Jeffreys, the need to define probability objectively is imposed by science itself. He aimed to define probability in a "pure" way suited for scientific applications. This led Jeffreys to criticize the subjective interpretation of probability put forward by Frank Ramsey, with whom he consorted and shared various interests but apparently never discussed probability.[25] To Jeffreys' eyes subjectivism is a "theory of expectation rather than one of "pure probability" [Jeffreys, 1936a, p. 326]. For a scientist like Jeffreys subjective probability is a theory "for business men". This is not meant as an expression of contempt, for "we have habitually to decide on the best course of action in given circumstances, in other words to compare the expectations of the benefits that may arise from different actions; hence a theory of expectation is possibly more needed than one of pure probability" [Jeffreys, 1939, 1961, p. 326]. But what science requires is a notion of "pure probability", not the subjective notion in terms of preferences based on expectations.

[24]For a discussion of the simplicity postulate see Howson [1988].

[25]According to Howie and Lindley, Jeffreys found out about Ramsey's work on probability only after Ramsey's death in 1930; see Howie [2002], p. 117 and Lindley [1991], p. 13. However, Howie provides evidence that both Ramsey and Jeffreys took part in a group discussing psychoanalysis, whose activity is described in Cameron and Forrester [2000]. Strangely enough, during those meetings they did not discuss probability.

In order to define probability in a "pure" way, Jeffreys grounds it on a principle, stated by way of an axiom, which says that probabilities are comparable: "given p, q is either more, equally, or less probable that r, and no two of these alternatives can be true" [Jeffreys, 1939, 1961, p. 16]. He then shows that the fundamental properties of probability functions follow from this assumption. By so doing, Jeffreys qualifies as one of the first to establish the rules of probability from basic presuppositions.

Although admitting an affinity with Keynes' perspective, Jeffreys is careful to keep his own position separate from that of Keynes. In the "Preface" to the second edition of his *Theory of Probability*, Jeffreys complains at having been labelled a "follower" of Keynes, and draws attention to the fact that Keynes' *Treatise on Probability* appeared after he and Dorothy Wrinch had published their first contributions to the theory of probability, drawing the lines of an epistemic approach akin to Keynes' logicism. He also points out that the resemblance between his own theory and that of Keynes depends on the fact that both attended the lectures of William Ernest Johnson [Jeffreys, 1939, 1961, p. v], thereby bringing more evidence of Johnson's influence on his contemporaries. A major disagreement with Keynes concerns Keynes' refusal "to admit that all probabilities are expressible by numbers" [Jeffreys, 1931, p. 223].[26] In that connection, Jeffreys' viewpoint coincides with subjectivism.

A most interesting aspect of Jeffreys' thought is that of developing an original epistemology, which is deeply probabilistic in character.[27] This is rooted in a phenomenalistic view of knowledge of the kind upheld by Ernst Mach and Karl Pearson. However, for Jeffreys "the pure phenomenalistic attitude is not adequate for scientific needs. It requires development, and in some cases modification, before it can deal with the problems of inference" [Jeffreys, 1931, p. 225]. The crucial innovation to be made with respect to Mach's phenomenalism amounts to the introduction of probability, or probabilistic inference, to be more precise. Jeffreys' epistemology is *constructivist*, in the sense that such crucial ingredients of scientific knowledge as the notions of "empirical law", "objectivity", "reality", and "causality" are established by inference from experience. This is made possible by statistical methodology, seen as the fundamental tool of science.

Concerning *objectivity*, in the "Addenda" to the 1937 edition of *Scientific Inference*, Jeffreys writes that "the introduction of the word 'objective' at the outset seems [...] a fundamental confusion. The whole problem of scientific method is to find out *what* is objective" [Jeffreys, 1931, 1973, p. 255]. The same idea is expressed in *Theory of Probability*, where he states: "I should query whether any meaning can be attached to "objective" without a previous analysis of the process of *finding out* what is objective" [Jeffreys, 1939, p. 336]. Such a process is inductive and probabilistic, it originates in our sensations and proceeds step by step to the construction of abstract notions lying beyond phenomena. Such notions

[26] Additional points of disagreement between Jeffreys and Keynes are described in Jeffreys [1922].

[27] This is described in more detail in Galavotti [2003].

cannot be described in terms of observables, but are nonetheless admissible and useful, because they permit "co-ordination of a large number of sensations that cannot be achieved so compactly in any other way" [Jeffreys, 1931, 1973, p. 190]. In this way *empirical laws*, or "objective statements", are established. To this end, an inductive passage is needed, for it is only after the rules of induction "have compared it with experience and attached a high probability to it as a result of that comparison" that a general proposition can become a law. In this procedure lies "the only scientifically useful meaning of 'objectivity"' [Jeffreys, 1939, p. 336].

Similar considerations apply to the notion of *reality*. According to Jeffreys, a useful notion of reality obtains when some scientific hypotheses receive from the data a probability which is so high, that on their basis one can draw inferences, whose probabilities are practically the same as if the hypotheses in question were certain. Hypotheses of this kind are taken as certain in the sense that all their parameters "acquire a permanent status". In such cases, we can assert the associations expressed by the hypotheses in question "as an approximate rule".

Jeffreys retains a likewise empirical and constructivist view of *causality*. His proposal is to substitute the general formulation of the "principle of causality" with "causal analysis", as performed within statistical methodology. This starts by considering all the variations observed in a given phenomenon at random, and proceeds to detect correlations which allow for predictions and descriptions that are the more precise, the better their agreement with observations. This procedure leads to asserting laws, which are eventually accepted because "the agreement (with observations) is too good to be accidental" [Jeffreys, 1937, p. 62]. Within scientific practice, the principle of causality is "inverted": "instead of saying that every event has a cause, we recognize that observations vary and regard scientific method as a procedure for analysing the variation" [Jeffreys, 1931, 1957, p. 78]. The deterministic version of the principle of causality is thereby discarded, for "it expresses a wish for exactness, which is always frustrated, and nothing more" [Jeffreys, 1937, pp. 63-64].

Jeffreys' position regarding scientific laws, reality and causality reveal the same pragmatical attitude underpinning Ramsey's views on general propositions and causality, the main difference being that Ramsey's approach is more strictly analytic, whereas Jeffreys grounds his arguments on probabilistic inference and statistical methodology alone. Furthermore, Jeffreys and Ramsey share the conviction that within an epistemic interpretation of probability there is room for notions like *chance* and *physical probability*. Jeffreys regards the notion of chance as the "limiting case" of everyday probability assignments. Chance occurs in those situations in which "given certain parameters, the probability of an event is the same at every trial, no matter what may have happened at previous trials" [Jeffreys, 1931, 1957, p. 46]. For instance, chance "will apply to the throw of a coin or a die that we previously know to be unbiased, but not if we are throwing it with the object of determining the degree of bias. It will apply to measurements when we know the true value and the law of error already. [...] It is not numerically assessable except when we know so much about the system already that we need to know no more"

[Jeffreys, 1936a, p. 329]. Jeffreys also contemplates the possibility of extending the realm of epistemic probability to a robust notion of "physical probability" of the kind encountered in quantum mechanics. He calls attention to those fields where "some scientific laws may contain an element of probability that is intrinsic to the system and has nothing to do with our knowledge of it" [Jeffreys, 1955, p. 284]. This is the case with quantum mechanics, whose account of phenomena is irreducibly probabilistic. Unlike the probability (chance) that a fair coin falls heads, *intrinsic* probabilities do not belong to our description of phenomena, but to the theory itself. Jeffreys claims to be "inclined to think that there may be such a thing as intrinsic probability. [...] Whether there is or not — he adds – it can be discussed in the language of epistemological probability" [Jeffreys, 1955, p. 284]. We will find similar ideas expressed by Ramsey.

The pragmatical attitude that characterizes Jeffreys' epistemology brings him close to subjectivism, and so does his conviction that science is fallible, together with his admission that empirical information can be "vague and half-forgotten", a fact that "has possibly led to more trouble than has received explicit mention" [Jeffreys, 1931, 1973, p. 406]. These features of his perspective are somewhat at odds with his definition of probability as a degree of rational belief uniquely determined by experience, and with the idea that the evaluation of probability is an objective procedure, whose application to experimental evidence obeys rules having the status of logical principles.

2 THE SUBJECTIVE INTERPRETATION OF PROBABILITY

Modern subjectivism, sometimes also called "personalism", shares with logicism the conviction that probability is an epistemic notion. As already pointed out, the crucial point of disagreement between the two interpretations comes in connection with the fact that unlike logicists, subjectivists do not believe that probability evaluations are univocally determined by a given body of evidence.

2.1 The starters

William Fishburn Donkin (1814–1869), professor of astronomy at Oxford, fostered a subjective interpretation of probability in "On Certain Questions Relating to the Theory of Probabilities", published in 1851. There he writes that "the 'probability' which is estimated numerically means merely 'quantity of belief', and is nothing inherent in the hypothesis to which it refers" [Donkin, 1851, p. 355]. This claim impressed Frank Ramsey, who recorded it in his notes.[28] Donkin's position is actually quite similar to that of De Morgan, especially when he maintains that probability is "*relative* to a particular state of knowledge or ignorance; but [...] it is *absolute* in the sense of not being relative to any individual mind; since, the

[28]See document 003-13-01 of the Ramsey Collection, held at the Hillman Library, University of Pittsburgh.

same information being presupposed, all minds *ought* to distribute their belief in the same way" [Donkin, 1851, p. 355]. If in view of claims of this kind Donkin qualifies more as a logicist than as a subjectivist, the appearance of his name in the present section on subjectivism is justified by the fact that he addressed the issue of *belief conditioning* in a way that anticipated the work of Richard Jeffrey a century later. Donkin formulated a principle, imposing a symmetry restriction on updating belief, as new information is obtained. In a nutshell, the principle states that changing opinion on the probabilities assigned to a set of hypotheses, after new information has been acquired, has to preserve the proportionality among the probabilities assigned to the considered options. Under this condition, the new and old opinions are comparable. The principle is introduced by Donkin as follows:

> "*Theorem.* If there be any number of mutually exclusive hypotheses, h_1, h_2, h_3 ..., of which the probabilities relative to a particular state of information are p_1, p_2, p_3..., and if new information be gained which changes the probabilities of some of them, suppose of h_{m+1} and all that follow, *without having otherwise any reference to the rest*, then the probabilities of these latter have the *same ratios* to one another, *after* the new information, that they had *before*; that is,
>
> $p'_1 \colon p'_2 \colon p'_3 \colon \ldots \colon p'_m = p_1 \colon p_2 \colon p_3 \colon \ldots \colon p_m$,
>
> where the accented letters denote the values after the new information has been acquired". [Donkin, 1851, p. 356]

The method of conditioning known as *Jeffrey conditionalization* reflects precisely the intuition behind Donkin's principle.[29]

The French mathematician Émile Borel (1871–1956), who gave outstanding contributions to the study of the mathematical properties of probability, can be considered a pioneer of the subjective interpretation. In a review of Keynes' *Treatise* originally published in 1924 and later reprinted in the last volume of the series of monographs edited by Borel under the title *Traité du calcul des probabilités et ses applications* (1939),[30] Borel raises various objections to Keynes, blamed for overlooking the applications of probability to science to focus only on the probability of judgments. Borel takes this to be a distinctive feature of the English as opposed to continental literature which he regards as more aware of the developments of science, particularly physics. When making such claims, Borel is likely to have in mind above all Henri Poincaré, whose ideas exercised a certain influence on him.[31]

[29] See Jeffrey [1965], [1992a] and [2004].

[30] The *Traité* includes 18 issues, collected in 4 volumes. The review of Keynes' *Treatise* appears in the last issue, under the title "Valeur pratique et philosophie des probabilités".

[31] See von Plato [1994, p. 36], where Borel is described as a successor of Poincaré "in an intellectual sense". The book by von Plato contains a detailed exposition of Borel's ideas on probability. See also [Knobloch, 1987].

While agreeing with Keynes in taking probability in its epistemic sense, Borel claims that probability acquires a different meaning depending on the context in which it occurs. Probability has a different value in situations characterized by a different state of information, and is endowed with a "more objective" meaning in science, where its assessment is grounded on a strong body of information, shared by the scientific community.

Borel is definitely a subjectivist when he admits that two people, given the same information, can come up with different probability evaluations. This is most common in everyday applications of probability, like horse races, or weather forecasts. In all such cases, probability judgments are of necessity relative to "a certain body of knowledge", which is not the kind of information shared by everyone, like scientific theories at a certain time. Remarkably, Borel maintains that when talking of this kind of probability the "body of knowledge" in question should be thought of as "necessarily included in a determinate human mind, but not such that the same abstract knowledge constitutes the same body of knowledge in two distinct human minds" [Borel, 1924, English edition 1964, p. 51]. Probability evaluations made at different times, based on different information, ought not be taken as refinements of previous judgments, but as totally new ones.

Borel disagrees with Keynes on the claim that there are probabilities which cannot be evaluated numerically. In connection with the evaluation of probability Borel appeals to the method of betting, which "permits us in the majority of cases a numerical evaluation of probabilities" [Borel, 1924, English edition 1964, p. 57]. This method, which dates back to the origin of the numerical notion of probability in the seventeenth century, is regarded by Borel as having

> "exactly the same characteristics as the evaluation of prices by the method of exchange. If one desires to know the price of a ton of coal, it suffices to offer successively greater and greater sums to the person who possesses the coal; at a certain sum he will decide to sell it. Inversely if the possessor of the coal offers his coal, he will find it sold if he lowers his demands sufficiently". [Borel, 1924, English edition 1964, p. 57]

At the end of a discussion of the method of bets, where he takes into account some of the traditional objections against it, Borel concludes that this method seems good enough, in the light of ordinary experience.

Borel's conception of epistemic probability has a strong affinity with the subjective interpretation developed by Ramsey and de Finetti. In a brief note on Borel's work, de Finetti praises Borel for holding that probability must be referred to the single case, and that this kind of probability is always measurable sufficiently well by means of the betting method. At the same time, de Finetti strongly disagrees with the eclectic attitude taken by Borel, more particularly with his admission of an objective meaning of probability, in addition to the subjective.[32]

[32] De Finetti's commentary on Borel is to be found in de Finetti [1939].

2.2 Ramsey and the principle of coherence

Frank Plumpton Ramsey (1903-1930), Fellow of King's College and lecturer in mathematics at Cambridge, made outstanding contributions to a number of different fields, including mathematics, logic, philosophy, probability, and economics.[33] In his obituary, Keynes refers to Ramsey's as "one of the brightest minds of our generation" and praises him for the "amazing, easy efficiency of the intellectual machine which ground away behind his wide temples and broad, smiling face" [Keynes, 1930, 1972, p. 336]. A regular attender at the meetings of the Moral Sciences Club and the Apostles, Ramsey actively interacted with his contemporaries, including Keynes, Moore, Russell and Wittgenstein — whose *Tractatus* he translated into English — often influencing their ideas.

Ramsey is considered the starter of modern subjectivism with his paper "Truth and Probability", read at the Moral Sciences Club in 1926, and published in 1931 in the collection *The Foundations of Mathematics and Other Logical Essays* edited by Richard Bevan Braithwaite shortly after Ramsey's death. Other sources are to be found in the same book, as well as in the other collection, edited by Hugh Mellor, *Philosophical Papers* (largely overlapping Braithwaite's), and in addition in the volumes *Notes on Philosophy, Probability and Mathematics*, edited by Maria Carla Galavotti, and *On Truth*, edited by Nicholas Rescher and Ulrich Majer.

Ramsey regards probability as a degree of belief, and probability theory as a logic of partial belief. *Degree of belief* is taken as a primitive notion having "no precise meaning unless we specify more exactly how it is to be measured" [Ramsey, 1990a, p. 63]; in other words, degree of belief requires an operative definition that specifies how it can be measured. A "classical" way of measuring degree of belief is the method of bets, endowed with a long-standing tradition dating back to the birth of probability in the seventeenth century with the work of Blaise Pascal, Pierre Fermat and Christiaan Huygens. In Ramsey's words: "the old established way of measuring a person's belief is to propose a bet, and see what are the lowest odds which he will accept" (Ramsey [1990a], p. 68). Such a method, however, suffers from well known problems, like the diminishing marginal utility of money, and is to a certain extent arbitrary, due to personal "eagerness or reluctance to bet", and the fact that "the proposal of a bet may inevitably alter" a person's "state of opinion" (Ramsey [1990a], p. 68).

To avoid such difficulties, Ramsey adopted an alternative method based on the notion of *preference*, grounded in a "general psychological theory" asserting that "we act in the way we think most likely to realize the objects of our desires, so that a person's actions are completely determined by his desires and opinions" [Ramsey, 1990a, p. 69]. Attention is called to the fact that

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[33]On Ramsey's life, see [Taylor, 2006] and the last chapter of [Sahlin, 1990]. See also "Better than the Stars", a radio portrait of Frank Ramsey written and presented by Hugh Mellor, with Alfred J. Ayer, Richard B. Braithwaite, Richard C. Jeffrey, Michael Ramsey (Archbishop of Canterbury and Frank's brother), Lettice Ramsey (Frank's widow), Ivor A. Richards, originally recorded in 1978, and later published in Mellor, ed. [1995]. More to be found in the Ramsey Archive of King's College, Cambridge.

> "this theory is not to be identified with the psychology of the Utilitarians, in which pleasure had a dominant position. The theory I propose to adopt is that we seek things which we want, which may be our own or other people's pleasure, or anything else whatever, and our actions are such as we think most likely to realize these goods." [Ramsey, 1990a, p. 69]

After clarifying that "good" and "bad" are not to be taken in an ethical sense, "but simply as denoting that to which a given person feels desire and aversion" [Ramsey, 1990a, p. 70], Ramsey introduces the notion of *quantity of belief*, by assuming that goods are measurable as well as additive, and that an agent "will always choose the course of action which will lead in his opinion to the greatest sum of good" [Ramsey, [990a, p. 70]. The fact that people hardly ever entertain a belief with certainty, and usually act under uncertainty, is accounted for by appealing to the principle of mathematical expectation, which Ramsey introduces "as a law of psychology". Given a person who is prepared to act in order to achieve some good,

> "if p is a proposition about which he is doubtful, any goods or bads for whose realization p is in his view a necessary and sufficient condition enter into his calculation multiplied by the same fraction, which is called the 'degree of his belief in p'. We thus define degree of belief in a way which presupposes the use of mathematical expectation". [Ramsey, 1990a, p. 70]

An alternative definition of degree of belief is also suggested along the following lines: "Suppose [the] degree of belief [of a certain person] in p **is** m/n; then his action is such as he would choose it to be if he had to repeat it exactly n times, in m of which p was true, and in the others false" [Ramsey, 1990a, p. 70]. The two accounts point out two different, albeit strictly intertwined, aspects of the same concept, and are taken to be equivalent.

Ramsey exemplifies a typical situation involving a choice of action that depends on belief as follows:

> "I am at a cross-roads and do not know the way; but I rather think one of the two ways is right. I propose therefore to go that way but keep my eyes open for someone to ask; if now I see someone half a mile away over the fields, whether I turn aside to ask him will depend on the relative inconvenience of going out of my way to cross the fields or of continuing on the wrong road if it is the wrong road. But it will also depend on how confident I am that I am right; and clearly the more confident I am of this the less distance I should be willing to go from the road to check my opinion. I propose therefore to use the distance I would be prepared to go to ask, as a measure of the confidence of my opinion". [Ramsey, 1990a, pp. 70-71]

Denoting $f\ (x)$ the disadvantage of walking x metres, r the advantage of reaching the right destination, and w the disadvantage of arriving at a wrong destination, if I were ready to go a distance d to ask, the degree of belief that I am on the right road is $p = 1 - (f(d)/(r - w))$. To choose an action of this kind can be considered advantageous if, were I to act n times in the same way, np times out of these n I was on the right road (otherwise I was on the wrong one). In fact, the total good of not asking each time is $npr + n(1 - p)w = nw + np(r - w)$; while the total good of asking each time (in which case I would never go wrong) is $nr - nf(x)$. The total good of asking is greater than the total good of not asking, provided that $f(x) \prec (r - w)(1 - p)$. Ramsey concludes that the distance d is connected with my degree of belief, p, by the relation $f(d) = (r - w)(1 - p)$, which amounts to $p = 1 - (f(d)/(r - w))$, as stated above. He then observes that

> "It is easy to see that this way of measuring beliefs gives results agreeing with ordinary ideas. [...] Further, it allows validity to betting as means of measuring beliefs. By proposing to bet on p we give the subject a possible course of action from which so much extra good will result to him if p is true and so much extra bad if p is false". [Ramsey, 1990a, p. 72]

However, given the already mentioned difficulties connected with the betting scheme, Ramsey turns to a more general notion of *preference*. Degree of belief is then operationally defined in terms of personal preferences, determined on the basis of the expectation of an individual of obtaining certain goods, not necessarily of a monetary kind. The value of such goods is intrinsically relative, because they are defined with reference to a set of alternatives. The definition of degree of belief is committed to a set of axioms, which provide a way of representing its values by means of real values. Degrees of belief obeying such axioms are called *consistent*. The laws of probability are then spelled out in terms of degrees of belief, and it is argued that consistent sets of degrees of belief satisfy the laws of probability. Additivity is assumed in a finite sense, since the set of alternatives taken into account is finite. In this connection Ramsey observes that the human mind is only capable of contemplating a finite number of alternatives open to action, and even when a question is conceived, allowing for an infinite number of answers, these have to be lumped "into a finite number of groups" [Ramsey, 1990a, p. 79].

The crucial feature of Ramsey's theory of probability is the link between probability and degree of belief established by *consistency*, or *coherence* — to use the term that is commonly adopted today. Consistency guarantees the applicability of the notion of degree of belief, which can therefore qualify as an admissible interpretation of probability. In Ramsey's words, the laws of probability can be shown to be

> "necessarily true of any consistent set of degrees of belief. Any definite set of degrees of belief which broke them would be inconsistent in the sense that it violated the laws of preference between options. [...] If anyone's mental condition violated these laws, his choice would depend

> on the precise form in which the options were offered him, which would be absurd. He could have a book made against him by a cunning better and would then stand to lose in any event.
>
> We find, therefore, that a precise account of the nature of partial belief reveals that the laws of probability are laws of consistency. [...] Having any definite degree of belief implies a certain measure of consistency, namely willingness to bet on a given proposition at the same odds for any stake, the stakes being measured in terms of ultimate values. Having degrees of belief obeying the laws of probability implies a further measure of consistency, namely such a consistency between the odds acceptable on different propositions as shall prevent a book being made against you". [Ramsey, 1990a, p. 78]

By arguing that from the assumption of coherence one can derive the laws of probability Ramsey paved the way to a fully-fledged subjectivism. Remarkably, within this perspective the laws of probability "do not depend for their meaning on any degree of belief in a proposition being uniquely determined as the rational one; they merely distinguish those sets of beliefs which obey them as consistent ones" [Ramsey, 1990a, p. 78]. This claim brings us to the core of subjectivism, for which coherence is the only condition that degrees of belief should obey, or, to put it slightly differently, insofar as a set of degrees of belief is coherent there is no further demand of rationality to be met.

Having adopted a notion of probability in terms of coherent degrees of belief, Ramsey does not need to rely on the principle of indifference. In his words: "the Principle of Indifference can now be altogether dispensed with" [Ramsey, 1990a, p. 85]. This is a decisive step in the moulding of modern subjectivism. As we will see in the next Section, a further step was made by Bruno de Finetti, who supplied the "static" definition of subjective probability in terms of coherent degrees of belief with a "dynamic" dimension, obtained by joining subjective probability with exchangeability within the framework of the Bayesian method.[34] Although this crucial step was actually made by de Finetti, there is evidence that Ramsey knew the property of exchangeability, of which he must have heard from Johnson's lectures. Evidence for this claim is found in his note "Rule of Succession", where use is made of the notion of exchangeability, named "equiprobability of all permutations".[35] What apparently Ramsey did not see, and was instead grasped by de Finetti, is the usefulness of applying exchangeability to the inductive procedure, modelled upon Bayes' rule. Remarkably, in another note called "Weight or the Value of Knowledge",[36] Ramsey was able to prove that collecting evidence pays in expectation, provided that acquiring the new information is free, and shows how much the increase in weight is. This shows he had a dynamic view at least of this

[34] This terminology is borrowed from Zabell [1991], containing useful remarks on Ramsey's contribution to subjectivism. For a comparison between Ramsey and de Finetti on subjective probability, see [Galavotti, 1991].

[35] See Ramsey [1991a], pp. 279-281. For a detailed commentary see Di Maio [1994].

[36] See Ramsey [1990b]; also included in [1991a, pp. 285-287].

important process. As pointed out by Nils-Eric Sahlin and Brian Skyrms, Ramsey's note on weight anticipates subsequent work by Savage, Good, and others.[37]

Ramsey put forward his theory of probability in open contrast with Keynes. In particular, Ramsey did not share Keynes' claim that "a probability may [...] be unknown to us through lack of skill in arguing from given evidence" [Ramsey, 1922, 1989, p. 220]. For a subjectivist, the notion of unknown probability does not make much sense, as repeatedly emphasized also by de Finetti. Moreover, Ramsey criticized the logical relations on which Keynes' theory rests. In "Criticism of Keynes" he writes that: "There are no such things as these relations. a) Do we really perceive them? Least of all in the simplest cases when they should be clearest; can we really know them so little and yet be so certain of the laws which they testify? [...] c) They would stand in such a strange correspondence with degrees of belief" [Ramsey, 1991a, pp. 273-274].

Like Keynes, Ramsey believed that probability is the object of logic, but they disagreed on the nature of that logic. Ramsey distinguished between a "lesser logic, which is the logic of consistency, or formal logic", and a "larger logic, which is the logic of discovery, or inductive logic" [Ramsey, 1990a, p. 82]. The "lesser" logic, which is the logic of tautologies in Wittgenstein's sense, can be "interpreted as an objective science consisting of objectively necessary propositions". By contrast, the "larger" logic, which includes probability, does not share this feature, because "when we extend formal logic to include partial beliefs this direct objective interpretation is lost" [Ramsey, 1990a, p. 83], and can only be endowed with a *psychological foundation*.[38] Ramsey's move towards psychologism was inspired by Wittgenstein. This is manifest in a paper read to the Apostles in 1922, called "Induction: Keynes and Wittgenstein", where Wittgenstein's psychologism is contrasted with Keynes' logicism. At the beginning of that paper, Ramsey mentions propositions 6.363 and 6.3631 of the *Tractatus*, where it is maintained that the process of induction "has no logical foundation but only a psychological one" [Ramsey, 1991a, p. 296]. After praising Wittgenstein for his appeal to psychology in order to justify the inductive procedure, Ramsey discusses Keynes' approach at length, expressing serious doubts on his attempt at grounding induction on logical relations and hypotheses. At the end of the paper, after recalling Hume's celebrated argument, Ramsey puts forward by way of a conjecture, of which he claims to be too tired "to see clearly if it is sensible or absurd", the idea that induction could be justified by saying that

> "a type of inference is reasonable or unreasonable according to the relative frequencies with which it leads to truth and falsehood. Induction is reasonable because it produces predictions which are generally verified, not because of any logical relation between its premisses and conclusions. On this view we should establish by induction that induction was reasonable, and induction being reasonable this would be a

[37] See Nils-Eric Sahlin's "Preamble" to Ramsey [1990b], and Skyrms [1990] and [2006]. See in addition Savage [1954] and Good [1967].

[38] For some remarks on Ramsey's psychological theory of belief see Suppes [2006].

> reasonable argument". [Ramsey, 1991a, p. 301]

This passage suggests that Ramsey had in mind a pragmatic justification of the inductive procedure. A similar attitude reappears at the end of "Truth and Probability", where he describes his own position as "a kind of pragmatism", holding that

> "we judge mental habits by whether they work, i.e. whether the opinions they lead to are for the most part true, or more often true than those which alternative habits would lead to.
>
> Induction is such a useful habit, and so to adopt it is reasonable. All that philosophy can do is to analyse it, determine the degree of its utility, and find on what characteristics of nature it depends. An indispensable means for investigating these problems is induction itself, without which we should be helpless. In this circle lies nothing vicious. It is only through memory that we can determine the degree of accuracy of memory; for if we make experiments to determine this effect, they will be useless unless we remember them". [Ramsey, 1990a, p. 93-94]

As testified by a number of Ramsey's references to William James and Charles Sanders Peirce, pragmatism is a major feature of his philosophy in general, and his views on probability are no exception.

A puzzling aspect of Ramsey's theory of probability are the relations between degree of belief and frequency. In "Truth and Probability" he writes that "it is natural [...] that we should expect some intimate connection between these two interpretations, some explanation of the possibility of applying the same mathematical calculus to two such different sets of phenomena" [Ramsey, 1990a, p. 83]. Such a connection is identified with the fact that "the very idea of partial belief involves reference to a hypothetical or ideal frequency [...] belief of degree m/n is the sort of belief which leads to the action which would be best if repeated n times in m of which the proposition is true" [Ramsey, 1990a, p. 84].

This passage — echoing the previously mentioned conjecture from "Induction: Keynes and Wittgenstein" — reaffirms Ramsey's pragmatical tendency to associate belief with action, and to justify inductive behaviour with reference to successful conduct. The argument is pushed even further when Ramsey says that

> "It is this connection between partial belief and frequency which enables us to use the calculus of frequencies as a calculus of consistent partial belief. And in a sense we may say that the two interpretations are the objective and subjective aspects of the same inner meaning, just as formal logic can be interpreted objectively as a body of tautology and subjectively as the laws of consistent thought". [Ramsey, 1990a, p. 84]

However, in other passages the connection between these two "aspects" is not quite so strict:

> "experienced frequencies often lead to corresponding partial beliefs, and partial beliefs lead to the expectation of corresponding frequencies in accordance with Bernoulli's Theorem. But neither of these is exactly the connection we want; a partial belief cannot in general be connected uniquely with any actual frequency". [Ramsey, 1990a, p. 84]

Evidence that Ramsey was intrigued by the relation between frequency and degree of belief is found in some remarks contained in the note "Miscellaneous Notes on Probability", written in 1928. There four kinds of connections are pointed out, namely: "(1) if degree of belief = γ, most prob((able)) frequency is γ (if instances independent). This is Bernoulli's theorem; (2) if freq((uency)) has been γ we tend to believe with degree γ; (3) if freq((uency)) is γ, degree γ of belief is justified. This is Peirce's definition; (4) degree γ of belief means acting appropriately to a frequency γ" [Ramsey, 1991a, p. 275]. After calling attention to such possible connections, Ramsey reaches the conclusion that "it is this last which makes calculus of frequencies applicable to degrees of belief". Remarkably, the result known as de Finetti's "representation theorem" tells us precisely how to treat relation (4). One might speculate that Ramsey would have found an answer to at least part of what he was looking for in this result, that de Finetti found out in the very same years, but was not available to him.[39]

Claims like that mentioned above to the effect that partial belief and frequency "are the two objective and subjective aspects of the same inner meaning", might be taken to suggest that Ramsey admitted of two notions of probability: one epistemic (the subjective view) and one empirical (the frequency view).[40] This emerges again at the very beginning of "Truth and Probability" where Ramsey claims that although the paper deals with the logic of partial belief, "there is no intention of implying that this is the only or even the most important aspect of the subject", adding that "probability is of fundamental importance not only in logic but also in statistical and physical science, and we cannot be sure beforehand that the most useful interpretation of it in logic will be appropriate in physics also" [Ramsey, 1990a, p. 53]. It can be argued that in spite of these claims Ramsey trusted that the subjective interpretation has the resources for accounting for all uses of probability. His writings offer plenty of evidence for this thesis.

There is no doubt that Ramsey took seriously the problem of what kind of probability is employed in science. We know from Braithwaite's "Introduction" to *The Foundations of Mathematics* that he had planned to write a final section of "Truth and Probability", dealing with probability in science. We also know from Ramsey's unpublished notes that by the time of his death he was working on a book bearing the title "On Truth and Probability", of which he left a number of tables of contents.[41] Of the projected book he only wrote the first part, dealing with the notion of truth, which was published in 1991 under the title *On Truth.* It can be conjectured that he meant to include in the second part of the book the

[39]On this point, see Galavotti [1991] and [1995].
[40]For instance, this opinion is upheld in Good [1965, p. 8].
[41]See the "Ramsey Collection" held by the Hillman Library of the University of Pittsburgh.

content of the paper "Truth and Probability", plus some additional material on probability in science. The notes published in *The Foundations of Mathematics* under the heading "Further Considerations",[42] and a few more published in the volume *Notes on Philosophy, Probability and Mathematics*, contain evidence that in the years 1928-29 Ramsey was actively thinking about such problems as theories, laws, causality, chance, all of which he regarded as intertwined. A careful analysis of such writings shows that — contrary to the widespread opinion that he was a dualist with regard to probability — in the last years of his life Ramsey was developing a view of chance and probability in physics fully compatible with his subjective interpretation of probability as degree of belief.

Ramsey's view of *chance* revolves around the idea that this notion requires some reference to scientific theories. Chance cannot be defined simply in terms of laws (empirical regularities) or frequencies — though the specification of chances involves reference to laws, in a way that will soon be clarified. In "Reasonable Degree of Belief" Ramsey writes that "We sometimes really assume a *theory* of the world with laws and chances and mean not the proportion of actual cases but what is chance on our theory" [Ramsey, 1990a, p. 97]. The same point is emphasized in the note "Chance", also written in 1928, where the frequency-based views of chance put forward by authors like Norman Campbell is criticized. The point is interesting, because it highlights Ramsey's attitude to frequentism, which, far from considering a viable interpretation of probability, he deems inadequate. As Ramsey puts it:

> "There is, for instance, no empirically established fact of the form 'In n consecutive throws the number of heads lies between $n/2 \pm \varepsilon(n)$'. On the contrary we have good reason to believe that any such law would be broken if we took enough instances of it.
>
> Nor is there any fact established empirically about infinite series of throws; this formulation is only adopted to avoid contradiction by experience; and what no experience can contradict, none can confirm, let alone establish". [Ramsey, 1990a, p. 104]

To Campbell's frequentist view, Ramsey opposed a notion of chance ultimately based on degrees of belief. He defines it as follows:

> "Chances are degrees of belief within a certain system of beliefs and degrees of belief; not those of any actual person, but in a simplified system to which those of actual people, especially the speaker, in part approximate. [...] This system of beliefs consists, firstly, of natural laws, which are in it believed for certain, although, of course, people are not really quite certain of them". [Ramsey, 1990a, p. 104]

In addition, the system will contain statements of the form: "when knowing ψx and nothing else relevant, always expect ϕx with degree of belief p (what is or

[42] In Ramsey [1931, pp. 199-211]. These are the notes called: "Reasonable Degree of Belief", "Statistics" and "Chance", all reprinted in [1990a, pp. 97-109].

is not relevant is also specified in the system)" [Ramsey, 1990a, p. 104]. Such statements together with the laws "form a deductive system according to the rules of probability, and the actual beliefs of a user of the system should approximate to those deduced from a combination of the system and the particular knowledge of fact possessed by the user, this last being (inexactly) taken as certain" [Ramsey, 1990a, p. 105]. To put it differently, chance is defined with reference to systems of beliefs that typically contain accepted laws.

Ramsey stresses that chances "must not be confounded with frequencies", for the frequencies actually observed do not necessarily coincide with them. Unlike frequencies, chances can be said to be "objective" in two ways. First, to say that a system includes a chance value referred to a phenomenon, means that the system itself cannot be modified so as to include a pair of deterministic laws, ruling the occurrence and non-occurrence of the same phenomenon. As explicitly admitted by Ramsey, this characterization of objective chance is reminiscent of Poincaré's treatment of the matter, and typically applies "when small causes produce large effects" [Ramsey, 1990a, p. 106]. Second, chances can be said to be objective "in that everyone agrees about them, as opposed e.g. to odds on horses" [Ramsey, 1990a, p. 106)].

On the basis of this general definition of chance, Ramsey qualifies *probability in physics* as chance referred to a more complex system, namely to a system making reference to scientific theories. In other words, probabilities occurring in physics are derived from physical theories. They can be taken as *ultimate chances*, to mean that within the theoretical framework in which they occur there is no way of replacing them with deterministic laws. The objective character of chances descends from the objectivity peculiarly ascribed to theories that are universally accepted.

Ramsey's view of chance and probability in physics is obviously intertwined with his conception of theories, truth and knowledge in general. Within Ramsey's philosophy the "truth" of theories is accounted for in pragmatical terms. In this connection Ramsey holds the view, whose paternity is usually attributed to Charles Sanders Peirce, but is also found in Campbell's work, that theories which gain "universal assent" in the long run are accepted by the scientific community and taken as true. Along similar lines he characterized a "true scientific system" with reference to a system to which the opinion of everyone, grounded on experimental evidence, will eventually converge. According to this pragmatically oriented view, chance attributions, like all general propositions belonging to theories — including causal laws — are not to be taken as propositions, but rather as "variable hypotheticals", or "rules for judging", apt to provide a tool with which the user meets the future.[43]

To sum up, for Ramsey chances are theoretical constructs, but they do not express realistic properties of "physical objects", whatever meaning be attached to this expression. Chance attributions indicate a way in which beliefs in various facts belonging to science are guided by scientific theories. Ramsey's idea that

[43]See especially "General Propositions and Causality" (1929) in Ramsey [1931] and [1990a].

within the framework of subjective probability one can make sense of an "objective" notion of physical probability has passed almost unnoticed. It is, instead, an important contribution to the subjective interpretation and its possible applications to science.

2.3 de Finetti and exchangeability

With the Italian Bruno de Finetti (1906-1985) the subjective interpretation of probability came to completion. Working in the same years as Ramsey, but independently, de Finetti forged a similar view of probability as degree of belief, subject to the only constraint of coherence. To such a definition he added the notion of exchangeability, which can be regarded as the decisive step towards the edification of modern subjectivism. In fact exchangeability, combined with Bayes' rule, gives rise to the inferential methodology which is at the root of the so-called neo-Bayesianism. This result was the object of the paper "Funzione caratteristica di un fenomeno aleatorio" that de Finetti read at the International Congress of Mathematicians, held in Bologna in 1928. In 1935, at Maurice Fréchet's invitation de Finetti gave a series of lectures at the Institut Henri Poincaré in Paris, whose text was published in 1937 under the title "La prévision: ses lois logiques, ses sources subjectives". This article, which is one of de Finetti's best known, allowed dissemination of his ideas in the French speaking community of probabilists. However, de Finetti's work came to be known to the English speaking community only in the 1950s, thanks to Leonard Jimmie Savage, with whom he entertained a fruitful collaboration. In addition to making a contribution to probability theory and statistics which is universally recognized as seminal, de Finetti put forward an original philosophy of probability, which can be described as a blend of pragmatism, operationalism and what we would today call "anti-realism".[44]indexAliotta, A.

Richard Jeffrey labelled de Finetti's philosophical position "radical probabilism"[45] to stress the fact that for de Finetti probability imbues the whole edifice of human knowledge, and that scientific knowledge is a product of human activity ruled by (subjective) probability, rather than truth or objectivity. De Finetti's outlined his philosophy of probability in the article "Probabilismo" (1931) which he regarded as his philosophical manifesto. Yet another philosophical text bearing the title *L'invenzione della verità*, originally written by de Finetti in 1934 to take part in a competition for a grant from the Royal Academy of Italy, was published in 2006. The two main sources of de Finetti's philosophy are Mach's phenomenalism, and pragmatism, namely the version upheld by the so-called Italian pragmatists, including Giovanni Vailati, Antonio Aliotta and Mario Calderoni. The starting point of de Finetti's probabilism is the refusal of the notion of truth, and the related view that there are "immutable and necessary" laws. In "Probabilismo" he

[44]This is outlined in some detail in Galavotti [1989]. For an autobiographical sketch of de Finetti's the reader is addressed to de Finetti [1982].
[45]See Jeffrey [1992b] and [1992c].

writes:

> "no science will permit us to say: this fact will come about, it will be thus and so because it follows from a certain law, and that law is an absolute truth. Still less will it lead us to conclude skeptically: the absolute truth does not exist, and so this fact might or might not come about, it may go like this or in a totally different way, I know nothing about it. What we can say is this: *I foresee* that such a fact will come about, and that it will happen in such and such a way, because past experience and its scientific elaboration by human thought make this forecast seem reasonable to me". [de Finetti, 1931a, English edition 1989, p. 170]

Probability makes forecast possible, and since a forecast is always referred to a subject, being the product of his experience and convictions, the instrument we need is the subjective theory of probability. For de Finetti probabilism is the way out of the antithesis between absolutism and skepticism, and at its core lies the subjective notion of probability. Probability "means degree of belief (as actually held by someone, on the ground of his whole knowledge, experience, information) regarding the truth of a *sentence*, or *event* E (a fully specified 'single' event or sentence, whose truth or falsity is, for whatever reason, unknown to the person)" [de Finetti, 1968, p. 45]. Of this notion, de Finetti wants to show not only that it is the only non contradictory one, but also that it covers all uses of probability in science and everyday life. This programme is accomplished in two steps: first, an operational definition of probability is worked out, second, it is argued that the notion of objective probability is reducible to that of subjective probability.

As we have seen discussing Ramsey's theory of probability, the obvious option to define probability in an operational fashion is in terms of betting quotients. Accordingly, the degree of probability assigned by an individual to a certain event is identified with the betting quotient at which he *would be* ready to bet a certain sum on its occurrence. The individual in question should be thought of as one in a condition to bet whatever sum against any gambler whatsoever, free to choose the betting conditions, like someone holding the bank at a gambling-casino. Probability is defined as the fair betting quotient he would attach to his bets. De Finetti adopts this method, with the proviso that in case of monetary gain only small sums should be considered, to avoid the problem of marginal utility. Like Ramsey, de Finetti states coherence as the fundamental and unique criterion to be obeyed to avoid a sure loss, and spells out an argument to the effect that coherence is a sufficient condition for the fairness of a betting system, showing that a coherent gambling behaviour satisfies the principles of probability calculus, which can be derived from the notion of coherence itself. This is known in the literature as the *Dutch book* argument.

It is worth noting that for de Finetti the scheme of bets is just a convenient way of making probability readily understandable, but he always held that there are other ways of defining probability. In "Sul significato soggettivo della probabilità" [de Finetti, 1931b], after giving an operational definition of probability in terms of coherent betting systems, de Finetti introduces a qualitative definition of subjective probability based on the relation of "at least as probable as". He then argues that it is not essential to embrace a quantitative notion of probability, and that, while betting quotients are apt devices for measuring and defining probability in an operational fashion, they are by no means an essential component of the notion of probability, which is in itself a primitive notion, expressing "an individual's psychological perception" [de Finetti, 1931b, English edition 1992, p. 302]. The same point is stressed in *Teoria delle probabilità*, where de Finetti describes the betting scheme as a handy tool leading to "simple and useful insights" [de Finetti, 1970, English edition 1975, vol. 1, p. 180], but introduces another method of measuring probability, making use of scoring rules based on penalties. Remarkably, de Finetti assigns probability an autonomous value independent from the notion utility, thereby marking a difference between his position and that of Ramsey and other supporters of subjectivism, like Savage.

The second step of de Finetti's programme, namely the reduction of objective to subjective probability, relies on what is known as the "representation theorem". The pivotal notion in this context is that of *exchangeability*, which corresponds to Johnson's "permutation postulate" and Carnap's "symmetry".[46] Summarizing de Finetti, events belonging to a sequence are *exchangeable* if the probability of h successes in n events is the same, for whatever permutation of the n events, and for every n and $h \leq n$. The representation theorem says that the probability of exchangeable events can be represented as follows. Imagine the events were probabilistically independent, with a common probability of occurrence p. Then the probability of a sequence e, with h occurrences in n, would be $p^h(1-p)^{n-h}$. But if the events are exchangeable, the sequence has a probability $P(e)$, represented according to de Finetti's representation theorem as a *mixture* over the $p^h(1-p)^{n-h}$ with varying values of p:

$$P(e) = \int_0^1 p^h (1-p)^{n-h} dF(p)$$

where the distribution function $F(p)$ is unique. The above equation involves two kinds of probability, namely the subjective probability $P(e)$ and the "objective" (or "unknown") probability p of the events considered. This enters into the mixture associated with the weights assigned by the function $F(p)$ representing a probability distribution over the possible values of p. Assuming exchangeability then amounts to assuming that the events considered are equally distributed and independent, given any value of p.

In order to understand de Finetti's position, it is useful to start by considering how an objectivist would proceed when assessing the probability of an unknown

[46] In his "farewell lecture", delivered at the University of Rome before his retirement, de Finetti says that the term "exchangeability" was suggested to him by Maurice Fréchet in 1939. Before adopting this terminology, de Finetti had made use of the term "equivalence". See de Finetti [1976, p. 283].

event. An objectivist would assume an objective success probability p. But its value would in general remain unknown. One could give weights to the possible values of p, and determine the weighted average. The same applies to the probability of a sequence e, with h successes in n independent repetitions. Note that because of independence it does not matter where the successes appear. De Finetti focuses on the latter, calling *exchangeable* those sequences where the places of successes do not make a difference in probability. These need not be independent sequences. An objectivist who wanted to explain subjective probability, would say that the weighted averages are precisely the subjective probabilities. But de Finetti proceeds in the opposite direction with his representation theorem: starting from the subjective judgment of exchangeability, one can show that there is only one way of giving weights to the possible values of the unknown objective probabilities. According to this interpretation, objective probabilities become useless and subjective probability can do the whole job. De Finetti holds that exchangeability represents the correct way of expressing the idea that is usually conveyed by the expression "independent events with constant but unknown probability". If we take an urn of unknown composition, says de Finetti, the above phrase means that, relative to each of all possible compositions of the urn, the events can be seen as independent with constant probability. Then he points out that

> "what is unknown here is the composition of the urn, not the probability: this latter is always known and depends on the subjective opinion on the composition, an opinion which changes as new draws are made and the observed frequency is taken into account". [de Finetti, 1995, English edition 2008, p. 163]

It should not pass unnoticed that for the subjectivist de Finetti probability, being the expression of the feelings of the subjects who evaluate it, is always definite and known.

From a philosophical point of view, de Finetti's reduction of objective to subjective probability is to be seen pragmatically; it follows the same pragmatic spirit inspiring the operational definition of subjective probability, and complements it. From a more general viewpoint, the representation theorem gives applicability to subjective probability, by bridging the gap between degrees of belief and observed frequencies. Taken in connection with Bayes' rule, exchangeability provides a model of how to proceed in such a way as to allow for an interplay between the information on frequencies and degrees of belief. By showing that the adoption of Bayes' method, taken in conjunction with exchangeability, leads to a convergence between degrees of belief and frequencies, de Finetti indicates how subjective probability can be applied to statistical inference.

According to de Finetti, the representation theorem answers Hume's problem because it justifies "why we are also intuitively inclined to expect that frequency observed in the future will be close to frequency observed in the past" [de Finetti, 1972a, p. 34]. De Finetti's argument is pragmatic and revolves around the task of induction: to guide inductive reasoning and behavior in a coherent way. Like

Hume, de Finetti thinks that it is impossible to give a logical justification of induction, and answers the problem in a psychologistic fashion.

De Finetti's probabilism is deeply Bayesian: to his eyes statistical inference can be entirely performed by exchangeability in combination with Bayes' rule. From this perspective, the shift from prior to posterior, or, as he preferred to say, from *initial* to *final* probabilities, becomes the cornerstone of statistical inference. In a paper entitled "Initial Probabilities: a Prerequisite for any Valid Induction" de Finetti takes a "radical approach" by which "all the assumptions of an inference ought to be interpreted as an overall assignment of initial probabilities" [de Finetti, 1969, p. 9]. The shift from initial to final probabilities receives a subjective interpretation, in the sense that it means going from one subjective probability to another, although objective factors, like frequencies, are obviously taken into account, when available.

As repeatedly pointed out by de Finetti, updating one's mind in view of new evidence does not mean changing opinion: "If we reason according to Bayes' theorem, we *do not* change our opinion. We keep the same opinion, yet updated to the new situation. If yesterday I was saying "It is Wednesday", today I would say "It is Thursday". However I have not changed my mind, for the day after Wednesday is indeed Thursday" [de Finetti, 1995, English edition 2008, p. 43]. In other words, the idea of correcting previous opinions is alien to his perspective, and so is the notion of a self-correcting procedure, retained by other authors, like Hans Reichenbach.

The following passage from the book *Filosofia della probabilità*, recently published in English under the title *Philosophical Lectures on Probability*, highlights de Finetti's deeply felt conviction that subjective Bayesianism is the only acceptable way of addressing probabilistic inference, and the whole of statistics. The passage also gives the flavour of de Finetti's incisive prose:

> "The whole of subjectivistic statistics is based on this simple theorem of calculus of probability [Bayes' theorem]. This provides subjectivistic statistics with a very simple and general foundation. Moreover, by grounding itself on the basic probability axioms, subjectivistic statistics does not depend on those definitions of probability that would restrict its field of application (like, e.g., those based on the idea of equally probable events). Nor, for the characterization of inductive reasoning, is there any need — if we accept this framework — to resort to empirical formulae. Objectivistic statisticians, on the other hand, make copious use of empirical formulae. The necessity to resort to them only derives from their refusal to allow the use of the initial probability. [...] they reject the use of the initial probability because they reject the idea that probability depends on a state of information. However, by doing so, they distort everything: not only as they turn probability into an objective thing [...] but they go so far as to turn it into a *theological* entity: they pretend that the 'true' probability exists, outside ourselves, independently of a person's own judgement".

[de Finetti, 1995, English edition 2008, p. 43]

For de Finetti objective probability is not only useless, but meaningless, like all metaphysical notions. This attitude is epitomized by the statement "probability does not exist", printed in capital letters in the "Preface" to the English edition of *Teoria delle probabilità*. A similar statement opens the article "Probabilità" in the *Enciclopedia Einaudi*: "Is it true that probability 'exists'? What could it be? I would say no, it does not exist" [de Finetti, 1980, p. 1146]. Such aversion to the ascription of an objective meaning to probability is a direct consequence of de Finetti's anti-realism, and is inspired by the desire to keep the notion of probability free from metaphysics.

Unfortunately, de Finetti's statement has fostered the feeling that subjectivism is surrounded by a halo of arbitrariness. Against this suspicion, it must be stressed that de Finetti's attack on objective probability did not prevent him from taking seriously the issue of objectivity. In fact he struggled against the "distortion" of "identifying objectivity and objectivism", deemed a "dangerous mirage" [de Finetti, 1962a, p. 344], but did not deny the problem of the objectivity of probability evaluations. To clarify de Finetti's position, it is crucial to keep in mind de Finetti's distinction between the *definition* and the *evaluation* of probability. These are seen by de Finetti as utterly different concepts which should not be conflated. To his eyes, the confusion between the definition and the evaluation of probability imprints all the other interpretations of probability, namely frequentism, logicism and the classical approach. Upholders of these viewpoints look for a unique criterion — be it frequency, or symmetry — and use it as grounds for both the definition and the evaluation of probability. In so doing, they embrace a "rigid" attitude towards probability, which consists "in defining (in whatever way, according to whatever conception) the probability of an event, and in univocally determining a function" [de Finetti, 1933, p. 740]. By contrast, subjectivists take an "elastic" attitude, according to which the choice of one particular function is not committed to a single rule or method: "the subjective theory [...] does not contend that the opinions about probability are uniquely determined and justifiable. Probability does not correspond to a self-proclaimed 'rational' belief, but to the effective personal belief of anyone" [de Finetti, 1951, p. 218]. For subjectivists there are no "correct" probability assignments, and all coherent functions are admissible. The choice of one particular function is regarded as the result of a complex and largely context-dependent procedure. To be sure, the *evaluation* of probability should take into account all available evidence, including frequencies and symmetries. However, it would be a mistake to put these elements, which are useful ingredients of the evaluation of probability, at the basis of its definition.

De Finetti calls attention to the fact that the evaluation of probability involves both objective and subjective elements. In his words: "Every probability evaluation essentially depends on two components: (1) the objective component, consisting of the evidence of known data and facts; and (2) the subjective component, consisting of the opinion concerning unknown facts based on known evidence' [de Finetti, 1974, p. 7]. The subjective component is seen as unavoidable, and for de Finetti the explicit recognition of its role is a prerequisite for the appraisal of objective elements. Subjective elements in no way "destroy the objective elements nor put them aside, but bring forth the implications that originate only after the conjunction of both objective and subjective elements at our disposal" [de Finetti, 1973, p. 366]. De Finetti calls attention to the fact that the collection and exploitation of factual evidence, the objective component of probability judgments, involves subjective elements of various kinds, like the judgment as to what elements are relevant to the problem under consideration, and should enter into the evaluation of probabilities. In practical situations a number of other factors influence probability evaluations, including the degree of competence of the evaluator, his optimistic or pessimistic attitudes, the influence exercised by most recent facts, and the like. Equally subjective for de Finetti is the decision on how to let belief be influenced by objective elements.

Typically, when evaluating probability one relies on information regarding frequencies. Within de Finetti's perspective, the interaction between degrees of belief and frequencies rests on exchangeability. Assuming exchangeability, whenever a considerable amount of information on frequencies is available this will strongly constrain probability assignments. But information on frequencies is often scant, and in this case the problem of how to obtain good probability evaluations becomes crucial. This problem is addressed by de Finetti in a number of writings, partly fruit of his cooperation with Savage.[47] The approach adopted is based on *penalty methods*, of the kind of the well known "Brier's rule". Scoring rules like Brier's are devised to oblige those who make probability evaluations to be as accurate as they can and, if they have to compete with others, to be honest. Such rules play a twofold role within de Finetti's approach. In the first place, they offer a suitable tool for an operational definition of probability, which is in fact adopted by de Finetti in his late works. In addition, these rules offer a method for improving probability evaluations made both by a single person and by several people, because they can be employed as methods for exercising "self-control", as well as a "comparative control" over probability evaluations [de Finetti, 1980, p. 1151].[48] The use of such methods finds a simple interpretation within de Finetti's subjectivism: "though maintaining the subjectivist idea that no fact can prove or disprove belief" — he writes — "I find no difficulty in admitting that any form of comparison between probability evaluations (of myself, of other people) and actual events may be an element influencing my further judgment, of the same status as any other kind of information" [de Finetti, 1962a, p. 360]. De Finetti's work in this connection is in tune with a widespread attitude, especially among Bayesian statisticians, that has given rise to a vast literature on "well-calibrated" estimation methods.

Having clarified that de Finetti's refusal of objective probability is not tantamount to a denial of objectivity, it should be added that such a refusal leads him to overlook notions like "chance" and "physical probability". Having embraced

[47] See Savage [1971] where such a cooperation is mentioned.
[48] For further details, the reader is addressed to Dawid and Galavotti [2009].

the pragmatist conviction that science is just a continuation of everyday life, de Finetti never paid much attention to the use made of probability in science, and held that subjective probability can do the whole job. Only the volume *Filosofia della probabilità* includes a few remarks that are relevant to the point. There de Finetti admits that probability distributions belonging to scientific theories — he refers specifically to statistical mechanics — can be taken as "more solid grounds for subjective opinions" [de Finetti, 1995, English edition 2008, p. 63]. This allows for the conjecture that late in his life de Finetti must have entertained the idea that probabilities encountered in science derive a peculiar "robustness" from scientific theories.[49] Unlike Ramsey, however, de Finetti did not feel the need to include in his theory a notion of probability specifically devised for application in science.

With de Finetti's subjectivism, the epistemic conception of probability is committed to a theory that could not be more distant from Laplace's perspective. Unsurprisingly, de Finetti holds that "the belief that the *a priori* probabilities are distributed uniformly is a well defined opinion and is just as specific as the belief that these probabilities are distributed in any other perfectly specified manner" [de Finetti, 1951, p. 222]. But what is more important is that the weaker assumption of exchangeability allows for a more flexible inferential method than Laplace's method based on independence. Last but not least, unlike Laplace de Finetti is not a determinist. He believes that in the light of modern science, we have to admit that events are not determined with certainty, and therefore determinism is untenable.[50] For an empiricist and pragmatist like de Finetti, both determinism and indeterminism are unacceptable, when taken as physical, or even metaphysical, hypotheses; they can at best be useful ways of describing certain facts. In other words, the alternative between determinism and indeterminism "is undecidable and (I should like to say) illusory. These are metaphysical diatribes over 'things in themselves'; science is concerned with what 'appears to us', and it is not strange that, in order to study these phenomena it may in some cases seem more useful to imagine them from this or that standpoint" [de Finetti, 1976, p. 299].

CONCLUDING REMARKS

The epistemic approach is a strong trend in the current debate on probability. Of the two interpretations that have been outlined, namely logicism and subjectivism, subjectivism seems by far more popular, at least within economics and more generally in the social sciences. This can be imputed to a number of reasons, the most obvious being that in the social sciences and economics personal opinions

[49] This is argued in some detail in Galavotti [2001] and [2005].

[50] The issue of determinism is addressed in de Finetti [1931c] and in the "Appendix" contained in de Finetti [1970]. Some comments on the de Finetti's attitude towards determinism are to be found in Suppes [2009] and Zabell [2009].

and expectations enter directly into the information used to support forecasts, forge hypotheses and build models.

The work of Ramsey and de Finetti has exercised a formidable influence on subsequent literature. Under the spell of their ideas, novel research fields have been explored, including the theory of decision and the so-called dynamics of belief developed by authors like Richard Jeffrey, Brian Skyrms and many others.[51] Equally impressive is the impact of Ramsey and de Finetti on the literature on exchangeability and Bayesian inference, with the work of L. J. Savage, I. J. Good, Dennis Lindley[52] and many others working in their wake. From a philosophical point of view, the pragmatism and pluralism characterizing the subjective approach, especially its insistence on the role of various contextual factors, including the individual judgment of experts in the evaluation of probability, have gained considerable consensus.

In the realm of natural sciences the prevailing tendency has always been to regard probability as an empirical notion and to assign it a frequentist interpretation. Exceptions to this tendency are the authors who have sided with logicism. One such exception is Harold Jeffreys, whose perspective was considered in Part I. In a similar vein, under the influence of Boole and Keynes the physicist Richard T. Cox derived the laws of probability from a set of postulates formulated in algebraic terms, introduced as *plausibility* conditions.[53] In addition, Cox investigated the possibility of relating probability to entropy, taken as a measure of information and uncertainty, an idea shared by another physicist, namely Edwin T. Jaynes. A strong supporter of Bayesianism and an admirer of Jeffreys' work, Jaynes put forward a "principle of maximum entropy" as an "objective" criterion for the choice of priors.[54]

The problem of suggesting *objective* criteria for the choice of prior probabilities is a burning topic within recent debate revolving around Bayesianism. This has given rise to a specific trend of research, labelled "objective Bayesianiam".[55] Work in this connection tends to transpose the fundamental divergence between logicism and subjectivism, which essentially amounts to the tenet shared by logicism but not subjectivism that a degree of belief should be univocally determined by a given body of evidence, to the framework of Bayesianism. It is on this ground that the influence of logicism on contemporary debates seems more tangible.

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[52] See Savage [1954], Good [1965] and [1983] and Lindley [1965].
[53] See Cox [1946] and [1961].
[54] See Jaynes [1983] and [2003].
[55] See Williamson [2009].

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