7

Imprecise Probabilities in Quantum Mechanics

STEPHAN HARTMANN

It is a pleasure to thank Patrick Suppes for his great support and for many years of stimulating discussions about a wide range of topics of mutual interest. The present project grew out of these discussions, and I look forward to work with him on it and other projects for many more years.

1 Introduction

In his entry on "Quantum Logic and Probability Theory" in the *Stanford Encyclopedia of Philosophy*, Alexander Wilce (2012) writes that "it is uncontroversial (though remarkable) that the formal apparatus of quantum mechanics reduces neatly to a generalization of *classical probability* in which the role played by a Boolean algebra of events in the latter is taken over by the *'quantum logic'* of projection operators on a Hilbert space." For a long time, Patrick Suppes has opposed this view (see, for example, the papers collected in Suppes and Zanotti (1996). Instead of changing the logic and moving from a Boolean algebra to a non-Boolean algebra, one can also 'save the phenomena' by weakening the axioms of probability theory and work instead with *upper and lower probabilities.* However, it is fair to say that despite Suppes' efforts upper and lower probabilities are not particularly popular in physics as well as in the foundations of physics, at least so far. Instead, quantum logics is booming again, especially since quantum information and computation became hot topics. Interestingly, however, imprecise probabilities are becoming more and more popular in formal epistemology as recent work by authors such as James Joyce (2010) and Roger White (2010) demonstrates.

In this essay I would like to give *one more reason* for the use of upper and lower probabilities in quantum mechanics and outline the research program that they inspire. The remainder of this essay is organized as follows. Sec. 2 introduces upper and lower probabilities. Sec. 3 turns to quantum mechanics and presents the CHSH inequality. We show that there is not always a joint probability distribution that reproduces observed quantum correlations. Sec. 4 argues that imprecise probabilities can be defined in these cases, and Sec. 5 concludes with a number of open questions.

Foundations and Methods from Mathematics to Neuroscience.
Colleen E. Crangle, Adolfo García de la Sienra and Helen Longino.

2 Imprecise Probabilities

Imprecise probabilities are well known from the theory of uncertain reasoning, Halpern (2005); Walley (1991). The starting point of the formal developments is the question of how to represent one's ignorance about a probability value. One way to do this is to introduce a lower probability measure P_* and an upper probability measure P^*, where the difference between the two is an agent's measure of her uncertainty about a probability assignment. To illustrate this, consider a coin tossing experiment and start with $P_*(\text{Heads}) = 0$ and $P^*(\text{Heads}) = 1$, which means that the agent is in a state of full uncertainty about the outcomes of the coin tossings. Then collect evidence and update $P_*(\text{Heads})$ and $P^*(\text{Heads})$ accordingly. If the coin is fair, then both measures will eventually converge to 1/2, i.e. the probability of a fair coin to land heads. Note that the use of uppers and lowers is compatible with the existence of a probability value. The uppers and lowers only express our uncertainty about the probability value.

Upper and lower probability measures are defined as follows Suppes and Zanotti (1996).

Definition 1 (Upper Probability). *Let Ω be a nonempty set, $\mathcal{B}$ a Boolean algebra on Ω, and P^* a real-valued function on $\mathcal{B}$. Then $\Omega = (\Omega, F, P^*)$ is an upper probability space if and only if for every A and B in $\mathcal{B}$, (i) $0 \leq P^*(A) \leq 1$, (ii) $P^*(\emptyset) = 0$ and $P^*(\Omega) = 1$, (iii) if $A \cap B = \emptyset$, then $P^*(A \cup B) \leq P^*(A) + P^*(B)$.*

Definition 2 (Lower Probability). *Let Ω be a nonempty set, $\mathcal{B}$ a Boolean algebra on Ω, and P_* a real-valued function on $\mathcal{B}$. Then $\Omega = (\Omega, F, P_*)$ is a lower probability space if and only if for every A and B in $\mathcal{B}$, (i) $0 \leq P^*(A) \leq 1$, (ii) $P_*(\emptyset) = 0$ and $P_*(\Omega) = 1$, (iii) if $A \cap B = \emptyset$, then $P_*(A \cup B) \geq P_*(A) + P_*(B)$.*

We also note the following definition:

Definition 3 (Upper-Lower Pair). *We call a pair (P_*, P^*) an upper-lower probability pair Ω, if for every A in $\mathcal{B}$ we have $P_*(A) \leq P^*(A)$.*

Note that lower probabilities are super-additive and upper probabilities are sub-additive, which has several consequences: First, the sum over all atoms of the algebra may lead to a value greater than 1 for uppers and smaller than 1 for lowers. Second, while for a probability measure $P(A) = \sum_{A',B,B'} P(A, A', B, B')$ holds, the following inequalities hold for uppers and lowers:

$$P^*(A) \rightarrow \sum_{A',B,B'} P^*(A, A', B, B')$$

$$P_*(A) \rightarrow \sum_{A',B,B'} P_*(A, A', B, B').$$

Interestingly, if *monotonicity* holds, then uppers and lowers are related in the following way: $P_*(A) = 1 - P^*(\overline{A})$, where $\overline{A}$ is the complement of A in $\mathcal{B}$. (We will see later that this relation does not hold in quantum mechanics.) For an interpretation of upper and lower probabilities in terms of betting odds, see Walley (1991).

3 Quantum Mechanics and the CHSH Inequality

Let us consider four binary random variables A, A', B and B' that can take the values $a_i, a'_i, b_i, b'_i = \pm 1$ for $i = 1, 2$. We assume *symmetry*, i.e. we only consider situations where $E(A) = E(A') = E(B) = E(B') = 0$ with the expectation value E defined in the usual way, i.e. $E(A) := \sum_{i=1}^{2} a_i \, p(a_i)$. Next, we define the quantity

$$\mathcal{F} := |E(AB) + E(AB') + E(A'B) - E(A'B')|, \qquad (1)$$

where the expectation value

$$E(AB) := \sum_{i,k=1}^{2} a_i \, b_k \, P(a_i, b_k) = \sum_{i,j,k,l=1}^{2} a_i \, b_k \, P(a_i, a'_j, b_k, b'_l) \qquad (2)$$

measures the *correlation* between the random variables A and B. P is a probability measure. Note that $E(AB)$ takes values in the interval $[-1, 1]$ and that these correlations can be measured. Generalizing Bell's theorem, Clauser et al. (1969) effectively showed the following.

Theorem 1. *If there is a joint probability distribution $P(A, A', B, B')$, then $\mathcal{F} \leq 2$* ("CHSH inequality").

The proof is in the appendix.

As is generally known, the CHSH inequality does not always hold. There are experimental setups that exhibit (quantum) correlations which violate the CHSH inequality. In experiments with correlated photons, for example, one can measure values of $\mathcal{F}$ up to $2\sqrt{2}$. These experiments starts with an EPR state of correlated photons, i.e. with the state $|\mathrm{EPR} >= 1/\sqrt{2} \cdot (|10> - |01>)$ where $|0>$ and $|1>$ represent the photon polarizations of the two subsystems **A** and **B**. One can then find measurement angles α and α' (at **A**) and β and β' (at **B**) such that the CHSH inequality is violated. Hence, there is not always a joint probability distribution over A, A', B and B' that reproduces the expectation values $E(AB)$ etc. Note that these expectation values can be calculated from quantum mechanics and that the experiments confirm the theory.

Let us now study the CHSH inequality for atoms. Experiments similar to the just-mentioned photon experiments can be performed with an EPR state of two 2-level atoms that are trapped in a cavity. Here $|0>$ and $|1>$ represent the states of a single 2-level atom being in the ground state or the excited state, respectively. Let $A := X_1$, $A' := Z_1$, $B := X_2 + Z_2$ and $B' := X_2 - Z_2$, where X_1 denotes the Pauli matrix σ_x applied to the state of subsystem 1. Z_1, X_2 etc. are defined accordingly. Note that *symmetry* holds i.e. $E(A) = E(A') = E(B) = E(B') = 0$. Next, we calculate $E(AB) = E(AB') = E(A'B) = -1/2\sqrt{2}$ and $E(A'B') = 1/2\sqrt{2}$. Hence $\mathcal{F} = 2\sqrt{2}$, i.e. the CHSH inequality is maximally violated.

Next, we examine what happens if the quantum state under consideration decays under the influence of decoherence Schlosshauer (2007). Clearly, how fast the state decays will depend on the experimental context. It is known, for example, that the decay is slower in a cavity than in free space. What is important to us is that if the EPR state decoheres, then the correlations in the system also decay and the CHSH inequality will eventually be satisfied after some time τ_0. Once the CHSH inequality is satisfied, the correlations can be explained classically, i.e. by a non-contextual local hidden variables model. Moreover, these correlations can then be accounted for by a joint probability distribution.

Let us now calculate the time τ_0 when this is the case. One way of modeling decoherence is by coupling the quantum system to a reservoir. One can then write down the Schrödinger equation for the system plus the reservoir (environment), make the Born-Markov approximation, trace out the environment and obtain a quantum master equation for the reduced state ρ of the system. ρ then satisfies the following quantum master equation, which is of the Lindblad form Breuer and Petruccione (2002):

$$\frac{d\rho}{dt} = -\frac{B}{2} \sum_{i=1}^{2} [\sigma_+^{(i)} \sigma_-^{(i)} \rho + \rho \sigma_+^{(i)} \sigma_-^{(i)} - 2\sigma_-^{(i)} \rho \sigma_+^{(i)}], \tag{3}$$

with the decay constant B. Using the theory of Generalized Dicke States, Hartmann (Forthcoming), this equation can be solved analytically. We then obtain for the time evolution of the initial state $\rho(0) = |\mathrm{EPR} >< \mathrm{EPR}|$:

$$\rho(\tau) = e^{-\tau} \rho(0) + (1 - e^{-\tau}) |00 >< 00|, \tag{4}$$

with $\tau := B\,t$.

Next, we calculate the expectation values of A, A', B and B' as defined above for a system in the state $\rho(\tau)$ and obtain:

$$< A >= 0, \quad < A' >=< B >= - < B' >= e^{-\tau} - 1 \tag{5}$$

To make sure that *symmetry* holds for all times τ, we replace $A \to \tilde{A} := A- < A >$ etc. Clearly, we then have $E(\tilde{A}) = E(\tilde{A}') = E(\tilde{B}) = E(\tilde{B}') = 0$. For the correlations, we obtain:

$$\begin{array}{ccc} < \tilde{A}\tilde{B} >=< AB > & , & < \tilde{A}'\tilde{B} >=< A'B > -(e^{-\tau} - 1)^2 \\ < \tilde{A}\tilde{B}' >=< AB' > & , & < \tilde{A}'\tilde{B}' >=< A'B' > +(e^{-\tau} - 1)^2 \end{array}$$

Next, we calculate $\tilde{\mathcal{F}}$ as a function of τ (see Eq. (1)). It is easy to see that a joint probability distribution over $\tilde{A}, \tilde{A}', \tilde{B}$ and $\tilde{B}'$ exists if $\tau > \tau_0 := 245$, i.e. after a relatively short period of time after the quantum state starts to decay (in units of the inverse decay constant B). Figure 1 shows $\tilde{\mathcal{F}}$ and, for comparison, also $\mathcal{F}$ as a function of τ, where $\mathcal{F}$ is calculated using the original operators A, A', B and B'.

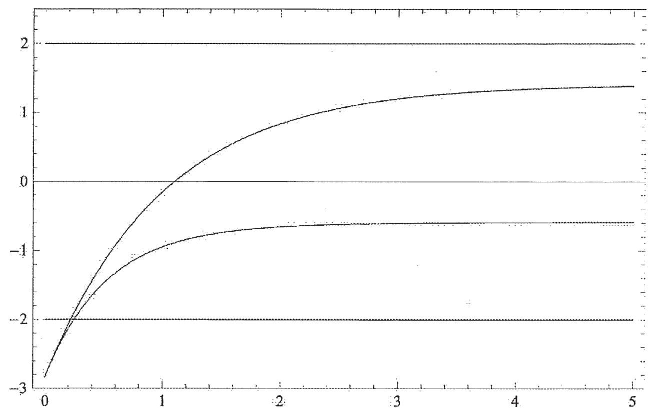

FIGURE 1 $\mathcal{F}$ (upper) and $\tilde{\mathcal{F}}$ (lower) as a function of τ

4 Imprecise Probabilities in Quantum Mechanics

We have seen that there is a joint probability distribution P for $\tau \geq \tau_0$ that reproduces the experimentally measurable correlations in the decaying EPR state. But how can we account for the correlations before that time? Hartmann and Suppes (2010) have explicitly constructed an upper probability distribution P^* that accounts for the correlations of a decaying EPR state at all times, i.e. before, at, and after τ_0. We therefore have unified account, which allows us to stick to a Boolean algebra of events. It is not necessary to work with a non-Boolean algebra in the quantum domain and a Boolean algebra in the classical domain, as quantum logicians do. All correlations can be accounted for by an upper probability distribution. This measure is explicitly sub-additive for times $\tau < \tau_0$ and turns into an additive probability measure for $\tau \geq \tau_0$. I take this to be a main advantage of the proposed approach to work with imprecise probabilities in quantum mechanics compared to the alternative quantum logical account, which do not allow for such a unified treatment.

It is interesting to note that the situation discussed here is similar to the learning situation discussed in Sec. 2. In the learning case, the upper probability distribution approximates the proper joint probability distribution more and more as the number of coin tosses increases. They coincide in the limit of an infinite number of coin tosses. In the quantum mechanical case, the upper probability distribution approximates the proper joint probability distribution more and more as the state decays. It coincides with the joint probability distribution once the CHSH inequality is satisfied (after a finite decay time). The joint probability distribution emerges from the interaction of the quantum state with its environment.

For the decaying EPR state, there is also a lower probability measure. This measure also converges into a probability measure which is defined for times $\tau \geq \tau_0$. However, the lower and the upper probability distributions are not related via $P_*(A) = 1 - P^*(\overline{A})$, i.e. they do not form an upper-lower pair. This is in line with the fact that there is no joint distribution for times $\tau < \tau_0$. Consequently, the *monotonicity* condition is violated in quantum mechanics, and upper and lower probability distributions have to be calculated independently by fitting them to the quantum mechanical expectation values. It is interesting to further explore the implications of the failure of monotonicity in quantum mechanics.

5 Open Questions

In future work, we plan to address the following four questions. *First*, how do our results generalize? Is it always possible, i.e. for all quantum states and corresponding sets of measurement operators, to fit an upper and a lower probability distribution? It would be nice to have a general proof that this is always possible, or a counter example showing that it is not. Our evidence so far is only episodic as we focused on the EPR state. *Second*, what is the proper interpretation of upper and lower probabilities in quantum mechanics? To address this question, the failure of *monotonicity* in quantum mechanics has to be understood. It will also be interesting to relate the discussion of upper and lower probabilities in quantum mechanics to the recent work on Quantum Bayesianism, Caves et al. (2007), which may shed some light on interpretational questions regarding upper and lower probabilities in quantum mechanics. *Third*, to further explore the relation between logic and probability in quantum mechanics, Gleason's Theorem has to be analyzed Hughes (1989). Here special attention has to be paid to the additivity assumption, which shows up in the proof of the theorem. We ask: What follows if one allows for sub- and super additive measures? *Fourth and finally*, what is the advantage of upper and lower probabilities compared to negative probabilities for which our decoherence story can be told as well? Negative probabilities were famously discussed by Feynman (1987) and have recently attracted the interest of Patrick Suppes. It will be worth to compare negative probabilities with imprecise probabilities.

Appendix: Proof of Theorem 1

To prove Theorem 1, we first simplify the notation and denote the value -1 by 0. Next, we introduce the following abbreviations:

$$\begin{aligned}
P(1111) = P(0000) &:= x_1 \quad , \quad P(1110) = P(0001) = x_2 \\
P(1101) = P(0010) &:= x_3 \quad , \quad P(1100) = P(0011) = x_4 \\
P(1011) = P(0100) &:= x_5 \quad , \quad P(1010) = P(0101) = x_6 \\
P(1001) = P(0110) &:= x_7 \quad , \quad P(1000) = P(0111) = x_8,
\end{aligned}$$

where we have made use of the *symmetry* requirement. Note that $0 \leq x_i \leq 1$ for $i = 1, \ldots, 8$ and that $\sum_{i=1}^{8} x_i = 1/2$. We then obtain by using Eq. (2) and similar equations for the other expectation values:

$$\begin{aligned}
\mathcal{F} &= 4|x_1 + x_2 - x_3 - x_4 + x_5 - x_6 + x_7 - x_8| \\
&\leq 4|x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8| \\
&\leq 4 \times 1/2 = 2,
\end{aligned}$$

which completes the proof.

References

Breuer, H. P. and F. Petruccione. 2002. *The Theory of Open Quantum Systems.* Oxford: Oxford University Press.

Caves, C. M., C. A. Fuchs, and R. Schack. 2007. Subjective probability and quantum certainty. *Studies in History and Philosophy of Modern Physics* 38(2):255–274. doi:10.1016/j.shpsb.2006.10.007.

Clauser, J. F., M. A. Horne, A. Shimony, and R. A. Holt. 1969. Proposed experiment to test local hidden-variable theories. *Physics Review Letters* 23(15):880–884.

Feynman, R. P. 1987. Negative probability. In B. J. Hiley and F. D. Peat, eds., *Quantum Implications : Essays in Honour of David Bohm*, pages 235–248. London: Routledge and Kegan Paul.

Halpern, J. 2005. *Reasoning about Uncertainty.* Cambridge, Mass.: MIT Press.

Hartmann, S. Forthcoming. Generalized Dicke states. ArXiv: quant-ph 1201.1732.

Hartmann, S. and P. Suppes. 2010. Entanglement, upper probabilities and decoherence in quantum mechanics. In M. Suarez, M. Dorato, and M. Redei, eds., *EPSA Philosophical Issues in the Sciences: Launch of the European Philosophy of Science Association*, vol. 2, pages 93–103. Berlin: Springer.

Hughes, R. I. G. 1989. *The Structure and Interpretation of Quantum Mechanics.* Cambridge, Mass.: Harvard University Press.

Joyce, J. M. 2010. A defense of imprecise credences in inference and decision making. *Philosophical Perspectives* 24(1):281–323. doi:10.1111/j.1520-8583.2010.00194.x.

Schlosshauer, M. A. 2007. *Decoherence and the Quantum-to-Classical Transition.* Berlin: Springer.

Suppes, P. and M. Zanotti. 1996. *Foundations of Probability with Applications.* Cambridge: Cambridge University Press.

Walley, P. 1991. *Statistical Reasoning with Imprecise Probabilities.* London: Chapman and Hall.

White, R. 2010. Evidential symmetry and mushy credences. In T. S. Gendler and J. Hawthorne, eds., *Oxford Studies in Epistemology*, vol. 3, pages 161–186. Oxford University Press.

Wilce, A. 2012. Quantum logic and probability theory. In *The Stanford Encyclopedia of Philosophy.* Fall 2012 edn. E. Zalta (ed.), URL `http://plato.stanford.edu/archives/fall2012/entries/qt-quantlog/`.

upon, not something to be proud of. I believe I stated my views on measurement in such a way that is easy to see where Adolfo and I agree, and where we do not.

Finally in Section 3, Adolfo discusses models of data in a way that I agree with on many points. But my effort has been to comment on where we disagree, and so this will be true of my final comment on this last section. In his comment on models of data, Adolfo emphasizes the distinction between finite data and the use of calculus and continuous quantities in the theory. I certainly agree that in any serious sense the data are finite, but the use of the calculus is really only accidental, as we see now so commonly in physics, where continuous theoretical quantities are mainly computed as numerically discrete, to meet the requirements of the computer programs doing the calculations. In fact, physics offers a wonderful example. The theory of approximation of many quantities has become so precise, and at the same time the mathematical difficulties of reasoning about these quantities in the standard form of classical mathematical analysis have become severe. These two developments have created a veritable revolution in how physical data are used in theory and in practice. The ever refined approximations of discrete data and computations now dominate physics, and so it will soon be in economics, as economists learn how to use all of the massive data that are available to them. Surely, economics is bound to become one of the "big data" sciences of the future.

8 Stephan Hartmann

I have known Stephan since he was a very young man in graduate school. We have had so many conversations about the foundations of quantum mechanics and related topics that I feel that this commentary is just a natural extension of that past. Stephan gives a clear and informative account of nonmonotonic upper probabilities as one appropriate framework for analyzing entanglement problems in quantum mechanics.

I want to add one supporting argument to the case he makes for the usefulness of such upper probabilities. It is not well enough recognized that the Hilbert-space formalism of von Neumann does not provide a framework for analyzing the individual sample paths of quantum particles. Rather, it is only mean probabilities that can be brought within the Hilbert-space formalism. Moreover, this formalism is not a natural one for studying the behavior of particles, even in the mean, over time. But, it is exactly temporal processing in decoherence that is central to a proper account. In contrast, the extension of the theory of nonmonotonic upper probabilities to stochastic processes is relatively straightforward, and can follow the lines of development of proper stochastic processes, so thoroughly studied in modern probability theory since the middle of the last century. I cannot help feeling that the collisions of particles that play such an important role in classical mechanics can be successfully entirely ignored in the temporal decoherence of quantum particles. This means I am predicting that the detailed theory of decoherence will need, in the end, to adopt the methods used in the analysis of stochastic processes, perhaps especially Brownian motion, with negative probabilities used in analysis of quantum entanglement. There is much more to be said on these matters, but this is not the right occasion.

I thoroughly agree with Stephan's choice of using the apparatus of upper probabilities in the framework of classical Boolean algebra. It is notable that the quantum logicians, by which I refer to those who work on quantum logic that is not classical, mention so seldom temporal processes, especially those of decoherence or decay. Somehow I feel it is much easier to accept a continuous sample path that at some point instantaneously loses its quantum character, for example, its quantum entanglement with another particle, rather than to think of a change of logic or algebra at this point. This is not the only argument, of course, to be considered in this debate, but it is one that occurs to me naturally and is consonant with the continued emphasis, even in quantum mechanics on temporal processing.

Finally, I like Stephan's open questions, and look forward to the solutions he develops in the future. Earlier in his article, Stephan mentions that physicists as yet have paid little attention to upper probabilities. No doubt within physics itself, their future is uncertain, but most surely they are an important and interesting alternative to be thoroughly explored.

9 R. Duncan Luce

I have known Duncan Luce, I believe, longer than anyone else contributing a paper to this volume. But this is only the beginning. We worked together on many projects, especially in the theory of measurement, until the time of his death in 2012. During this long period, he moved around a great deal and was on the faculty of several different universities at different times, but most of his time was spent either at Harvard or UC Irvine. I have talked to him so much and so often about the theory of measurement that I almost started to formulate a question for him about another paper in this volume. This is no longer possible, but I cherish our many years of work together and still have vivid memories of the times we spent and worked together, including traveling and dining in many places. I dedicate my commentary to him, but with a certain irony, for I know my way of looking at measurement was not a favorite of his. I do emphasize that this is only a particular viewpoint about a particular problem in the theory of measurement on which we were not entirely in agreement.

My aim is to mitigate any sense of disagreement, by showing how my own viewpoint supplements, but does not contradict anything Duncan has to say in his generalization of Hölder's Theorem. I take the general problem of measurement for both of us, and many others, has been concern to justify that a given scientific quantity, such as mass or velocity in physics, or utility in economics or subjective probability in psychology, satisfies (i) certain structural qualitative axioms, (ii) a quantitative representation theorem, (iii) a unique group of transformations defining the quantity's scale type, and (iv) invariance of the scientific quantity under this group of transformations. There is no doubt that the article Duncan contributed here and others he has written are a significant addition to the literature on Hölder's Theorem and the more general problem I defined.

My own approach is to relate the formal properties of a scientific quantity more directly to the theories that use it. In spite of my dedication to empiricism in the philosophy of science and recognizing the great importance of experimentation, structural questions are usually dominated by general theory, not a particular theory of measurement. My proposal is one that I made much earlier in a paper written with J.C.C. McKinsey (1953). The basic idea is straightforward, but can easily lead to the complicated proofs. Characterize the set of all models that can map models of the theory into models of the theory. The main focus of investigation is to determine the formal properties of these mappings or, in other language, transformations. For example, what transformations carry models of classical mechanics into models of classical mechanics. I use the word "transformations" loosely, because in this setting a transformation of the entire system is an n-tuple of transformations of scientific quantities that are not independent, but ordinarily strongly related, by the laws of the given theory. Each quantity,

CSLI Lecture Notes Number 213

Foundations and Methods from Mathematics to Neuroscience

Essays Inspired by Patrick Suppes

edited by
Colleen E. Crangle, Adolfo García de la Sienra, and Helen E. Longino

CSLI PUBLICATIONS
STANFORD

Center for the Study of Language and Information
Leland Stanford Junior University
Printed in the United States
19 18 17 16 15   1 2 3 4 5

Library of Congress Cataloging-in-Publication Data

Crangle, Colleen, 1952–.
Foundations and methods from mathematics to neuroscience / Colleen Crangle, Adolfo Garcia de la Sienra, and Helen Longino.
p. cm.
"Center for the Study of Language and Information, Leland Stanford Junior University."
Includes bibliographical references.
ISBN 978-1-57586-744-1 (cloth : alk. paper) –
ISBN 978-1-57586-745-8 (pbk. : alk. paper) –
ISBN 978-1-57586-746-5 (electronic)
1. Science–Philosophy. 2. Science–Philosophy–History–20th century. 3. Logic, Symbolic and mathematical. 4. Suppes, Patrick, 1922–2014. I. García de la Sienra, Adolfo. II. Longino, Helen E. III. Center for the Study of Language and Information (U.S.) IV. Title.

Q175.C8825 2015
501–dc23

2015000906

CIP

∞ The acid-free paper used in this book meets the minimum requirements of the American National Standard for Information Sciences—Permanence of Paper for Printed Library Materials, ANSI Z39.48-1984.

CSLI was founded in 1983 by researchers from Stanford University, SRI International, and Xerox PARC to further the research and development of integrated theories of language, information, and computation. CSLI headquarters and CSLI Publications are located on the campus of Stanford University.

CSLI Publications reports new developments in the study of language, information, and computation. Please visit our web site at
http://cslipublications.stanford.edu/
for comments on this and other titles, as well as for changes and corrections by the author and publisher.